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Solution for System of Fractional Order Wiener-Hope Dynamical System and System of Nonlinear Variational Inequality Problem

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Abstract—*The aim of this paper is to introduce a new system of Wiener - Hopf equation (SWHE) defined on a real Hilbert space. We study the system of nonlinear variational inequality problem on real Hilbert space. we consider a system of new fractional order Wiener-Hopf dynamical system (SFOWHDS) for system of nonlinear variational inequalities problem (SNVIP) using the Wiener-Hopf equations technique. Moreover, the existence of a solution to such a fractional order Wiener -Hopf dynamical system is considered and there is demonstrated a systemic solution to such a dynamical system. We show that the solution of system of fractional order Wiener-Hopf dynamical system is exist and unique. This type dynamical system is interesting to study because it can be apply in the various real world problems.*

Keywords: *Variational inequality problem, fractional derivative, Wiener- Hope equation, projected dynamical system, Lipschitz continuous mapping, non-expansive mapping, exponentially stability.*

INTRODUCTION

Integer order differential and integral equations (IDEs) make up the majority of the mathematical models. Since a few decades ago, non-integer order differential equations (FDEs) have allowed for the more accurate and precise formulation of actual events. Many researchers have grown passionate in the study of fractional differential system dynamics in recent years, and many interesting and significant outcomes, which include factional-order differential systems having chaos have been reported. Recently, For the purpose of learning to use fractional calculus, Nonlinear system stability analysis has been enhanced. The use of fractional calculus to model nonlinear systems served as an inspiration, these studies used the integer-order stabilisation approach.

The direct approach of fractional Lyapunov are suggested by the author in an effort to extend our understanding of fractional calculus and system theory. The use of fractional calculus in reality is made practical and inexpensive by quicker processing and less expensive memory. [Chen, [8]]. There are various area like informatics and material, control of fractional order dynamical system. In some cases, a fractional-order controller for a non-integer order system may perform better in terms of transient response than a traditional integer-order controller. Modern calculus is the generalization of classical integer-order calculus. Important uses in the sciences of mechanics, viscoelasticity, signal processing, economics, optimization, oceanography, bacteria that randomly move through fractal materials in search of food, neurons modelling, chaotic systems and others as well. It is significant to highlight that fractional differential systems can be used to explain a wide range of physical phenomena that include memory and inherited characteristics. For more read, we go to references [9]- [12].

In 2014, Zeng at.al. [14] studies at a class of global non-integer order projective dynamical systems and demonstrating the existence and originality of this kind of system's solution. With regard to these dynamical systems, it is possible to establish whether the equilibrium point exists and with the suitable conditions, its α-exponential stability.

Stampacchia initially proposed the variational inequality problem in 1964 [1], whose definition is as below:

Let C be a non-empty subset of Hilbert space H which is closed and convex and let consider nonlinear mapping T from subset C to H. The typical VIP is then introduced in the manner described below:

$\langle T(x^*), x - x^* \rangle \geq 0$,for all$x \in C$. (1.1)

Variational inequality problem (VIP) is the name given to equation (1.1) and indicated by VI(C,T) and the collection of all solution of (1.1) is indicated by Ω(VI(C,T)), that is,

$$\Omega(VI(C,T)) = \{x^* \in C : \langle T(x^*), x - x^* \rangle \geq 0, \forall x \in C\}.$$

The collection of all T's fixed point is indicated by Fix(T). It is well know results that VIP (1.1), which is outlined as the fixed point problem (FPP) that follows:

findx^*in$C$such that$x^* = P_C(I - \mu T)x^*$. (1.2)

where P_C is refer best approximation operator. from Hilbert space H to C, where $\mu > 0$ is non-negative constant and I stand for mapping from H on to H, which is identity. If the mapping T is η-strongly monotone and κ-Lipschitzian, then the operator $P_C(I - \mu T)$ is a contraction on subset C if $0 < \mu < 2\eta/\kappa^2$. The Banach Contraction principle in this situation ensures that equation (1.1) has exactly one solution x^* in C. Sequence is described as

$x_{n+1} = P_C(I - \mu T)x_n, \forall n \in \mathbb{N}$, (1.3)

converges x^* in C is known as The Picard iteration method's. This process is also called projection gradient method (PGM) (see [2]). Stampacchia studied the problem of variational inequality which widely use in field of mechanics. Moreover variational inequality is a one of the power full tool to studying different problem which are related to different branches of pure and applied mathematics. It is very useful in field of differential equation mechanics, transportation problem, operation research, control problem, equilibrium problem, fuzzy controls system and networking related problem. many authors use the concept of projection gradient method (PGM) in different ways, (see [3] [15] [16] [17]). This all technique are used in diverse area of science and being productive and innovative. This tech- nique are motivate to generalized the problem and extends the concept of variational inequality and convex optimization problem.

In 2001, Verma [17] presented the generalized variational inequality problem system, which studied as below:

Let $T: H \to H$ be the operator be nonlinear and C be a convex and closed subset of Hilbert space H that is not empty, to find $x^*, y^* \in C$,such that

$$\begin{cases} \langle \rho T(x^*) + y^* - x^*, x - y^* \rangle & \geq 0, \text{for all} x \in C, \\ \langle \eta T(y^*) + x^* - y^*, x - x^* \rangle & \geq 0, \text{for all} x \in C. \end{cases} \quad (1.4)$$

here ρ, η > 0 be constant. In 2001, Verma [17] Some algorithmic methods involving converges analysis for roughly addressing the VIP has been proposed. Convex optimisation problems and various other linear and nonlinear variational inequality problems are resolved as well using the projected dynamical system (for more information, see [18, 20]). In 1993, D. Zhang and A. Nagurney [21] introduce the Dynamical system and Variational inequality problem both and further in 1996 Further they studied about Projected dynamical system and VIP and provide some important results.

Noor [24] investigated the fixed point formulation in 2003 for Evaluation of the differential equation for quasi type VIP is the goal. Numerous dynamical systems recognised and proposed by Dupuis and Nagurney [21] are included in this dynamical system and Friesz et al. [23].

Cojocaru et al. [4] in 2005, A Lipschitz continuous operator on every Hilbert space of finite dimensional, for any nonempty convex and compact set, evolutionary projected dynamical systems, and variational inequality problem were explored, and they demonstrated the solution to this sort of problem.

The topic of dynamical systems has drawn the interest of several authors, who have written in these publications as a consequence of their thorough investigation. (see [21] [26] [28] [20] and the references therein).

On the other hand, In 1991, P. Shi [16] introduced the Wiener-Hopf equation and In 1992, Robinson [35] also studied Wiener-Hopf equation independently and using the projection technique. Wiener - Hope equation define as follows:

Consider no-void, closed and convex subset C of real Hilbert space H and T be a nonlinear operator from C to H, We view that problem as finding $x \in H$ such that

$Q_C x + \rho T P_C x = 0$, (1.5)

where $\rho > 0$ be constant and $Q_C = I - P_C$ substantiate the Wiener-Hope equation's equality with the variational inequalities. This show that solution of Wiener-Hope equation and solution of variational inequality problem can obtain if one of them exist and also unique. In 1993, Noor [36] show that generalized Wiener - Hope equation is equivalent to the variational inequity problem. In 2002, Noor [38] established the Wiener-Hopf equations method to analyse a dynamical system for variational inequality and to demonstrate the dynamical system's global asymptotic stability. The Wiener - Hopf dynamical system has global asymptotically stability property for pseudomonotone operator. In 2007, Noor and Zhenyu Huang consider about the types of nonlinear and non-expansive operators utilised by the new class of Wiener Hopf equations.In 2010, Guanghui Gu and Yongfu Su [39] studied approximations of the Wiener-Hopf equation and generalised variational inequality problem. In 2013, Changun Wu [40] give theory to find the solution of Wiener - Hopf equation and common solution of variational inequality and sand a collection of non-expansive mapping's fixed points under some condition.

Section 2 of this paper offers preliminary information, while Section 3 contains the major fact which show the solution of system of fractional order Wiener Hopf Projected dynamical system are exist and unique. Section 4 contains conclusion of this paper.

PRELIMINARIES

Firstly, we introduce some definition and lemma which are useful.

Definition 2.1 A nonlinear operator T from C to H is called

(1) monotone if

$$\langle Tx - Ty, x - y \rangle \geq 0, \textit{for all} x, y \in C,$$

(2) η-strongly monotone if $\exists\, \eta > 0$ such that

$\langle Tx - Ty, x - y\rangle \geq \eta \parallel x - y \parallel^2$,for all $x, y \in C$,

(3) β-Lipschitzian if $\exists\, \beta > 0$ such that

$\parallel Tx - Ty \parallel \leq \beta \parallel x - y \parallel$,for all $x, y \in C$,

(4) Non-expansive if

$\parallel Tx - Ty \parallel \leq \parallel x - y \parallel$,for all $x, y \in C$,

(5) Contraction if ∃ k ∈ [0, 1) s. t.

$\parallel Tx - Ty \parallel \leq k \parallel x - y \parallel$,for all $x, y \in C$.

Let $x \in H$ be a element not belong to subset C. A point $z \in C$ is said to be a nearest point to x if $d(x, C) = \parallel x - y \parallel$. The set of all best approximations from x to C, which may or may not be empty, is denoted by

$P_C(x) = \{y \in C: d(x, C) = \parallel x - y \parallel\}$ (2.1)

Consider the closed, convex, nonempty subset C of the set H. Then, for any element $x \in H$, there exist a unique best approximation point (nearest point) $P_C(x)$ of C such that

$\parallel x - P_C(x) \parallel \leq \parallel x - y \parallel$,for all $y \in C$. (2.2)

Note that P_C is non-expansive from H onto C. Lemma 2.1 [29] Given $x \in H$, $z \in C$, Then $P_C(x) = z$ if and only if $\langle x - z, z - y\rangle \geq 0$,for all $y \in C$.

Proposition 2.1. [30] For any element $x \in C$ and any $v \in C$ the limit $\prod_C(x, v) = \lim_{\delta \to 0^+} \frac{P_C(x+\delta v)-x}{\delta}$,

exits and $\prod_C(x, v) = P_C(v)$.

Definition 2.3. [30] Let H be a Hilbert space having any possible dimensions and $C \subseteq H$ be a closed, convex, and nonempty subset. Let F be only one-valued mapping on C. Then the differential equation

$\frac{dx(t)}{dt} = \prod_C(x(t), -F(x(t))), x(0) = x_0 \in C$ (2.3)

is said to be the F and C-related projected differential equation. Then a solution to (2.3) is x(t) an absolutely continuous function if $x: [0, T) \subseteq R \to H$ with $x(t) \in C$, ∀ t∈ $[0, T)$ and $dx/dt = \prod_C(x(t), -F(x(t)))$, for almost every $t \in [0, T)$.

To corroborate our conclusions regarding the concepts of stability in dynamic systems, the following definitions and lemma are important.

Take note of the general differential equation

$\frac{dx}{dt} = f(x(t))$, (2.4)

Definition 2.5 [31]

If $f(x^*) = 0$ then x^* point is referred to as the equilibrium point of equation (2.4).

If for any $\varepsilon > 0$, $\exists \delta > 0$ such that, for any $x_0 \in B(x^*, \delta)$ the solution x(t) of the differential equation with initial point $x(0) = x_0$ exists and $x(t) \in B(x^*, \varepsilon)$ (2.5) for all t > 0, then an equilibrium point x^* of (2.4) called stable.

Lemma 2.2. [33] (Gronwall Lemma) Let $\hat{u}$ and $\hat{v}$ be continuous real-valued functions with a domain $\{t: t \geq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 be monotone non-decreasing function. If for all $t \geq t_0$,

$\hat{u}(t) \leq \alpha(t) + \int_{t_0}^{t} \hat{u}(s)\hat{v}(s)ds$. (2.6)

Then $\hat{u}(t) \leq \alpha(t)e^{\int_{t_0}^{t} \hat{v}(s)ds}$. (2.7)

Lemma 2.3. [38] The VIP (1.1) have solution $x^* \in C$ iff the Wiener - Hopf equation (1.5) have unique solution $u \in H$ where

$x = P_C u$, (2.8)

$u = x - \rho Tx$. (2.9)

Using the equation (2.8) and (2.9), the Wiener - Hope equation can be written as

$x - \rho Tx - P_C[x - \rho Tx] + \rho TP_C[x - \rho Tx] = 0$. (2.10)

Using above equivalence, Noor analyze a new system associate with VIP (1.1) as follows:

$\frac{dx}{dt} = \lambda\{P_C[x - \rho Tx] - \rho TP_C[x - \rho Tx] + \rho Tx - x\}$.

(2.11)

with $x(t_0) = x_0$ and λ is constant. Equation (2.11) is known as Wiener - Hopf dynamical system.

Now, let's think about new dynamical system:

$$D_\omega^\alpha u(t) = \gamma\{P_C(u(t) - \rho Tu(t)) - \rho TP_C(u(t) - \rho Tu(t)) + \rho T(u(t)) - u(t)\},$$

(2.12)

where $\alpha \in (0,1)$ and γ is a constant related to VIP (1.1). The system (2.12) is called fractional order Wiener-Hopf dynamical system (FOWHDS) associated with a VIP (1.1).

Definition 2.9. [45] Riemann-Liouville definition of non-integer derivative of order $\alpha \in \mathbb{R}$, of u(t) is described as:

$I_{t_0}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \tau)^{\alpha-1} u(\tau)d\tau, t > t_0$, (2.13)

where the Euler gamma function is denoted by Γ.

Definition 2.10. [45] The Caputo derivative of non-integer derivative of order $\alpha \in \mathbb{R}_+$ of function $u(t) \in \mathbb{C}^n, ([t_0, +\infty], \mathbb{R})$ is given by

$$D_{t_0}^{\alpha}u(t) = I_{t_0}^{n-\alpha}u^{(n)}(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}u^{(n)}(\tau)d\tau, t > t_0,$$

(2.14)

where n is an integer that is positive so that $\alpha \in (n-1, n)$.

Definition 2.12. [47] If the dynamical system's (2.12) any two solutions u(t) and v(t) with distinct beginning point by u_0 and v_0 satisfy the condition

$$\| u(t) - v(t) \| \leq \eta \| u_0 - v_0 \| e^{-\lambda t \alpha}, \forall t \geq t_0,$$

then the system (2.12) is called α-exponentially stable with degree λ.

Lemma2.4 [48] Let $n \in \mathbb{Z}_+$ and $n-1 < \alpha < n$. $u(t) \in \mathbb{C}^n[a,b]$, then

$$I_t^{\alpha} D_t^{\alpha} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^k(a)}{k!}(t-a)^k.$$

In particular, if $0 < \alpha < 1$ and $u(t) \in C^1[a,b]$. $I_t^{\alpha} D_t^{\alpha} u(t) = u(t) - u(a)$. (2.15)

Lemma 2.5. [47] Consider a function, which is a continuous on $[0, +\infty)$ and satisfies

$D_t^{\alpha} u(t) \leq \theta u(t)$, (2.16)

where $0 < \alpha < 1$ and θ is a constant. Then

$$u(t) \leq u(0).exp\left(\frac{\theta t^{\alpha}}{\Gamma(\alpha+1)}\right).$$

Lemma 2.6. [47]- [49] Consider the system

$D_t^{\alpha} u(t) = g(t, u(t)), t > t_0$, (2.17)

with in initial condition $u(t_0)$, where $0 < \alpha \leq 1$and $g: [t_0, \infty) \times C \to H, C \subset H$. If $g(t, u(t))$ be locally Lipschitz continuous with regard to u(t), then ∃ unique solution of (2.17) on $[t_0, \infty) \times C$.

Lemma2.7. [49] With respects to the continuous function with real values g(t , u(t)), mentioned in (2.17), we have

$$\| I_t^{\alpha} g(t, u(t)) \| \leq I_t^{\alpha} \| g(t, u(t)) \|,$$

Where $\alpha \geq 0$ and $\|\cdot\|$ indicates an arbitrary norm.

MAIN RESULTS

First we discuss about some important Lemma and results: In 2001, Verma [17] present following lemma:

Lemma 3.1. [17] Solution of problem (1.4) are x^* and y^* iff

$y^* = P_C(x^* - \rho T x^*)$ and $x^* = P_C(y^* - \eta T y^*)$, (3.1)

where ρ, η be a positive constant.

The VIP (1.4), is similar to the system of Wiener-Hopf equations is now being considered. Let T be a nonlinear mapping from Hilbert space H to itself and $\rho, \eta > 0$ be constant, we regard this problem to be to identify x^*, y^*, u^*, v^* in H such that

$$\begin{cases} Q_C(v^*) + \rho T P_C(u^*) & = y^* - x^*, \\ Q_C(u^*) + \eta T P_C(v^*) & = x^* - y^*, \end{cases} \quad (3.2)$$

Where $Q_C = I - P_C$ where I be an identity operator on H.

In 2018, Narin Petrot and Jittiporn Tangkhawiwetkul [42] present the lemma, which show the equivalence of the problem (1.4) and (3.2).

Lemma 3.2. [42] Let $T: H \to H$ be a continuous Lipschitz mapping. There are solutions to the system of VIP (1.4) as $x^*, y^* \in C$ iff the system of equation (3.2) has solutions $x^*, y^*, u^*, v^* \in H$, where

$$\begin{cases} x^* & = P_C(v^*), \\ y^* & = P_C(u^*), \end{cases} \quad (3.3)$$

$$\begin{cases} u^* & = x^* - \rho T x^*, \\ v^* & = y^* - \eta T y^*. \end{cases} \quad (3.4)$$

We suggest the following generalized system of fractional order Wiener - Hope Dynamical system as follows:

$$\begin{cases} D_t^{\alpha} x(t) & = \lambda_1\{P_C(y - \eta T(y)) - \eta T P_C(x - \rho T(x)) + \eta T(y) - x\}, \\ D_t^{\alpha} y(t) & = \lambda_2\{P_C(x - \rho T(x)) - \rho T P_C(y - \eta T(y)) + \rho T(x) - y\}, \end{cases}$$

(3.5)

which $x(t_0), y(t_0)$ in C, λ_1, λ_2 are constant with real positive t_0.

Theorem 3.3. Let C be the real Hilbert space H's closed and convex subset, which is non-empty. Consider a Lipschitz continuous mapping T with constant β from H to H. Then, for each $x_0, y_0 \in H$, generalized system of fractional order Wiener - Hope Dynamical system (3.5) has the exactly one continuous solutions, x(t), and y(t) with $x(t_0) = x_0$ and $y(t_0) = y_0$ over $[t_0, \infty)$.

Proof. let λ_1, λ_2 are two constants and the mapping G from cartesian product $H \times H$ to itself, define as follow:

$$G(x, y) = (f(x), h(y)),$$

where

$f(x) = \lambda_1\{P_C(y - \eta T(y)) - \eta T P_C(x - \rho T(x)) + \eta T(y) - x\}$,and

$h(y) = \lambda_2\{P_C(x - \rho T(x)) - \rho T P_C(y - \eta T(y)) + \rho T(x) - y\}$,

for all x and y in H. We may now specify the norm $\|\cdot\|_1$ on $H \times H$ by

$$\| (x, y) \|_1 = \| x \| + \| y \|, \forall (x, y) \in H \times H. \quad (3.6)$$

We known that $H \times H$ is a Hilbert space in regard to the norm $\|\cdot\|_1$. First, G is a Lipschitz continuous mapping, as we shall demonstrate. For this let $(x_1, y_1), (x_2, y_2) \in H \times H$. We have

$\| G(x_1, y_1) - G(x_2, y_2) \|_1 = = \| (f(x_1), h(y_1)) - (f(x_2), h(y_2)) \|_1$

$= \| (f(x_1) - f(x_2), h(y_1) - h(y_2)) \|_1$

$= \| f(x_1) - f(x_2) \| + \| h(y_1) - h(y_2) \|$

$= \| \lambda_1\{P_C(y_1 - \eta T(y_1)) - \eta T P_C(x_1 - \rho T(x_1)) + \eta T(y_1) - x_1\} - (\lambda_1\{P_C(y_2 - \eta T(y_2)) - \eta T P_C(g(x_2) - \rho T(x_2)) + \eta T(y_2) - x_2\}) \| + \| \lambda_2\{P_C(x_1 - \rho T(x_1)) - \rho T P_C(y_1 - \eta T(y_1)) + \rho T(x_1) - y_1\} - (\lambda_2\{P_C(x_2 - \rho T(x_2)) - \rho T P_C(y_2 - \eta T(y_2)) + \rho T(x_2) - y_2\}) \|$

$= \lambda_1 \| P_C(y_1 - \eta T(y_1)) - \eta(\mu F - T P_C(x_1 - \rho T(x_1)) + \eta T(y_1) - x_1 - P_C(y_2 - \eta T(y_2)) + \eta T P_C(x_2 - \rho T(x_2)) - \eta T(y_2) + x_2 \| + \lambda_2 \| P_C(x_1 - \rho T(x_1)) - \rho T P_C(y_1 - \eta T(y_1)) + \rho T(x_1) - y_1 - P_C(x_2 - \rho T(x_2)) + \rho T P_C(y_2 - \eta T(y_2)) - \rho T(x_2) + y_2 \|$

$\leq \lambda_1\{\| P_C(y_1-\eta(\mu F-T(y_1))-P_C(y_2-\eta T(y_2)) \| +\eta \| TP_C(g(x_1)-\rho T(x_1))-TP_C(x_2-\rho T(x_2)) \| +\eta \| T(y_1)-T(y_2) \| + \| x_1-x_2 \|\} +\lambda_2\{\| P_C(x_1-sT(x_1))-P_C(x_2-\rho T(x_2)) \| +\rho \| TP_C(y_1-\eta T(y_1))-TP_C(y_2 -\eta(y_2)) \| +\rho \| T(x_1)-T(x_2) \| + \| y_1-y_2 \|\}$

$\leq \lambda_1\{\| y_1-\eta T(y_1)-(y_2-\eta T(y_2)) \| +\eta\beta \| P_C(x_1-\rho T(x_1))-(P_C(x_2-\rho T(x_2))) \| +\eta\beta \| y_1-y_2 \| + \| x_1-x_2 \|\}+\lambda_2\{\| x_1-\rho T(x_1)-(x_2-\rho T(x_2)) \| +\rho\beta \| P_C(y_1-\eta T(y_1))-(y_2-\eta T(y_2)) \| +\rho\beta \| x_1-x_2 \| + \| y_1-y_2 \|\}$

$\leq \lambda_1\{\| y_1-y_2 \| +\eta\beta \| y_1-y_2 \| +\eta\beta\{\| x_1-x_2 \| +\rho\beta \| x_1-x_2 \|\}+\eta\beta \| y_1-y_2 \| + \| x_1-x_2 \|\}+\lambda_2\{\| x_1-x_2 \| +\rho\beta \| x_1-x_2 \| +\rho\beta\{\| y_1-y_2 \| +\eta\beta \| y_1-y_2 \|\}+\eta\beta \| x_1-x_2 \| + \| y_1-y_2 \|\}$

$\leq \lambda_1\{(1+2\eta\beta) \| y_1-y_2 \| +(1+\eta\beta+\eta\rho\beta^2) \| x_1-x_2 \|\}+\lambda_2\{(1+2\rho\beta) \| x_1-x_2 \| +(1+\rho\beta+\eta\rho\beta^2) \| y_1-y_2 \|\}$

$\leq \lambda_1\{(1+2\Phi\beta) \| y_1-y_2 \| +(1+\Phi\beta+\Phi^2\beta^2) \| x_1-x_2 \|\}+\lambda_2\{(1+2\Phi\beta) \| x_1-x_2 \| +(1+\Phi\beta+\Phi^2\beta^2) \| y_1-y_2 \|\} \leq \Delta\{(1+2\Phi\beta) \| y_1-y_2 \| +(1+\Phi\beta+\Phi^2\beta^2) \| x_1-x_2 \|\}+\Delta\{(1+2\Phi\beta) \| x_1-x_2 \| +(1+\Phi\beta+\Phi^2\beta^2) \| y_1-y_2 \|\} = \Delta(1+2\Phi\beta)\{\| x_1-x_2 \| + \| y_1-y_2 \|\}+\Delta(1+\Phi\beta+\Phi^2\beta^2)\{\| x_1-x_2 \| + \| y_1-y_2 \|\} \leq 2\Delta(1+2\Phi\beta+\Phi^2\beta^2)\{\| x_1-x_2 \| + \| y_1-y_2 \|\} = 2\Delta(1+\Phi\beta)^2\{\| x_1-x_2 \| + \| y_1-y_2 \|\} = 2\Delta(1+\Phi\beta)^2\{ \| (x_1-x_2, y_1-y_2) \|_1\},$

where $\Delta = max\{\lambda_1,\lambda_2\}, \Phi = max\{\rho,\eta\}$. Then G is Lipschitz continuous on$\|\cdot\|_1$. Hence for each point $(x_0,y_0) \in H\times H$, system (3.5) has precisely one continuous solution (x(t), y(t)), defined on $t\in[t_0,\Gamma)$ with the initial conditions $x(t_0)=x_0$ and $y(t_0)=y_0$.

Let $[t_0,\Gamma)$ be the maximum period of existence. Now, we prove that $\Gamma=\infty$. Under the assumptions made of T, the VIP (1.4) has unique solution, $x^*,y^*\in C$, with $x^*=P_C(y^*-\eta T(y^*)), y^*=P_C(x^*-\rho T(x^*))$,

Let x and y be arbitrary element of Hilbert space H. Then, we have

$\| G(x,y) \|_1 = \| (f(x),h(y)) \|_1 = \| f(x) \| + \| h(y) \| = \| \lambda_1\{P_C(y-\eta T(y))-\eta TP_C(x-\rho T(x))+\eta T(y)-x\} \| + \| \lambda_2\{P_C(x-\rho T(x))-\rho(TP_C(y-\eta T(y))+\rho T(x)-y\} \| = \lambda_1\{\| P_C(y-\eta T(y))-x \| +\eta \| T(y)-TP_C(x-\rho T(x)) \|\}+\lambda_2\{\| P_C(x-\rho T(x))-y \| +\rho \| T(x)-TP_C(y-\eta T(y)) \|\}$

$\leq \lambda_1 \| P_C(y-\eta T(y))-x \| +\lambda_1\eta\beta \| y-(P_C(x-\rho T(x))) \| +\lambda_2 \| P_C(x-\rho T(x))-y \| +\lambda_2\rho\beta \| x-(P_C(y-\eta T(y))) \|$

$= (\lambda_1+\lambda_2\rho\beta) \| P_C(y-\eta T(y))-x \| +(\lambda_2+\lambda_1\eta\beta) \| P_C(x-\rho T(x))-y \|$

$\leq (\Delta+\Delta\Phi\beta)\{\| P_C(y-\eta T(y))-x \| + \| P_C(x-\rho T(x))-y \|\}$

$\leq (\Delta+\Delta\Phi\beta)\{\| P_C(y-\eta T(y))-P_C(y*-\eta T(y*)) \| + \| P_C(y*-\eta T(y*))-x^* \| + \| x^*-x \| + \| P_C(x-\rho T(x))-P_C(x^*-\rho T(x^*) \| + \| P_C(x^*-\rho T(x^*))-y^* \| + \| y^*-y \|\}$

$\leq (\Delta+\Delta\Phi\beta)\{\| x^*-x \| + \| y^*-y \| + \| y-\eta T(y)-(y^*-\eta T(y^*)) \| + \| x-\rho T(x)-(x^*-\rho T(x^*)) \|\} \leq (\Delta+\Delta\Phi\beta)\{ \| x-x^* \| + \| y-y^* \| + \| y-y^* \| +\Phi\beta \| y-y^* \| + \| x-x^* \| +\Phi\beta \| x-x^* \|\}$

$= (\Delta+\Delta\Phi\beta)(2+\Phi\beta)\{\| x-x^* \| + \| y-y^* \|\} = (\Delta+\Delta\Phi\beta)(2+\Phi\beta)\{\| x \| + \| x^* \| + \| y \| + \| y^* \|\} = (\Delta+\Delta\Phi\beta)(2+\Phi\beta)\{\| x^* \| + \| y^* \|\}+(\Delta+\Delta\Phi\beta)(2+\Phi\beta)\{\| x \| + \| y \|\} = (\Delta+\Delta\Phi\beta)(2+\Phi\beta) \| (x^*,y^*) \|_1 + (\Delta+\Delta\Phi\beta)(2+\Phi\beta) \| (x,y) \|_1,$

Hence,

$\| D_\omega^\alpha(x(t),y(t)) \| = \| G(x,y) \|_1 \leq k_1+k_2 \| (x,y) \|_1$, (3.7)

where, $k_1=(\Delta+\Delta\Phi\beta)(2+\Phi\beta) \| (x^*,y^*) \|_1$ and $k_2=(\Delta+\Delta\Phi\beta)(2+\Phi\beta)$. Taking the fractional integral of (3.7), we get

$I_\omega^\alpha \| D_\omega^\alpha(x(t),y(t)) \| \leq I_\omega^\alpha[k_1+k_2 \| (x,y) \|_1], \leq \frac{k_1}{\Gamma(\alpha)}\int_{t_0}^t(t-\tau)^{\alpha-1}d\tau+\frac{k_2}{\Gamma(\alpha)}\int_{t_0}^t(t-\tau)^{\alpha-1} \| (x(\tau),y(\tau)) \|_1 d\tau = \frac{k_1(t-t_0)^\alpha}{\Gamma(\alpha+1)}\frac{k_2}{\Gamma(\alpha)}\int_{t_0}^t(t-\tau)^{\alpha-1} \| (x(\tau),y(\tau)) \|_1$ (3.8)

Using Lemma 2.4 & 2.7, we get

$\| (x(t),y(t)) \| \leq \left\{\| (x(t_0),y(t_0)) \| + \frac{k_1(t-t_0)^\alpha}{\Gamma(\alpha+1)}\right\} + \frac{k_2}{\Gamma(\alpha)}\int_{t_0}^t(t-\tau)^{\alpha-1} \| (x(\tau),y(\tau)) \|_1 d\tau \leq \left\{\| (x(t_0),y(t_0)) \| + \frac{k_1(t-t_0)^\alpha}{\Gamma(\alpha+1)}\right\} exp\left\{\frac{k_2(t-t_0)^\alpha}{\Gamma(\alpha+1)}\right\}$,(3.9)

Hence, from (3.9) , Consequently, the solution is bounded on $[t_0,\infty)$. Therefore solution of generalize system of Wiener-Hopf dynamical system (3.5) is bounded on interval $[t_0,\Gamma)$, if Γ is finite. So, As a result, we say that $\Gamma=\infty$. Hence system of generalized fractional order Wiener-Hopf dynamical system (3.5) has exactly one continuous solution, x(t), y(t) with $x(t_0)=x_0$ and $y(t_0)=y_0$ over $[t_0,\Gamma]$. This complete the proof.

CONCLUSION

For the conventional system of variational inequalities, we have introduced and analysed the system of non-integer order Wiener-Hopf dynamical systems. The projection approach is devised and used to analyse these system of fractional dynamical systems connected to the system of variational inequalities. Under certain acceptable conditions, we have demonstrated that these fractional order Wiener-Hopf dynamical systems have only one solution to a system of VIP. Recurrent neural networks can be designed using the described dynamical systems to address variational inequalities and associated optimisation issues. Another potential direction for future work is to observe the stability of system of non-inter order Wiener-Hopf resolvent dynamical system and its application.

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Souvenir

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प्रकाशक

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Need and Application of Technology in Modern Education System

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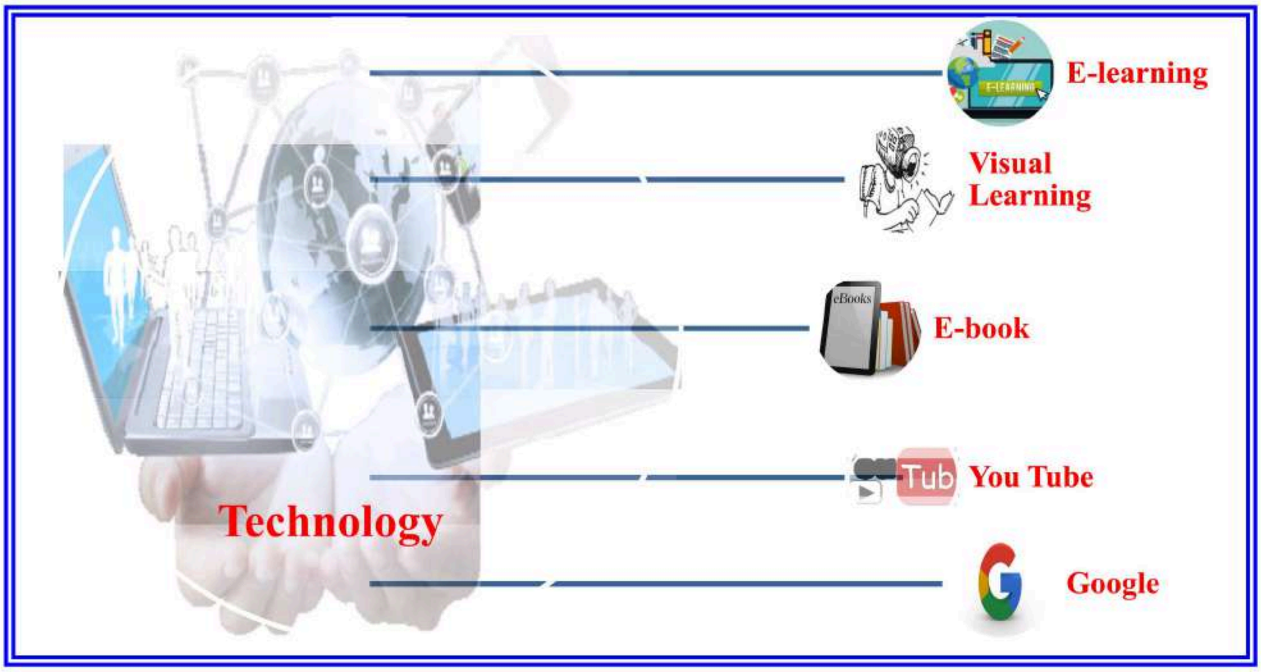
Graphical Abstract

This chapter explores the numerous ways that technology is being used in the field of education, such as e-learning platforms, virtual classrooms, educational apps, and online assessment tools. It highlights the advantages of these programs, including their adaptability to different learning styles and scalability to meet the demands of different sized classes.



Abstract

It is now essential for the 21st century educational system to include technology in order to satisfy the changing demands of both students and teachers. In this chapter, the vital role that technology plays in education is examined, along with a wide range of its uses, with a focus on how it has a profoundly positive influence on both teaching and learning. As a result of technology improvements, the educational scene has undergone major changes recently. A more dynamic, interactive, and individualized teaching method made possible by technology is progressively replacing the conventional chalk and board method. The need to better engage students, improve their educational experiences, and get them ready for a future that is largely

digital is what drives the demand for technology in education.This chapter explores the numerous ways that technology is being used in the field of education, such as e-learning platforms, virtual classrooms, educational apps, and online assessment tools. It highlights the advantages of these programs, including their adaptability to different learning styles and scalability to meet the demands of different sized classes. It looks at how technology has democratized education by giving students everywhere access to high-quality resources and teachers, promoting cross-cultural knowledge exchange and cooperation. It demonstrates how technology is changing the face of education and offers a thorough overview of the opportunities and difficulties that come with its integration, highlighting the significance of careful and responsible implementation to realize technology's full potential for the advantage of both students and educators.

Keywords: Technology, Dynamic, Chalk, Board, E-learning, Tools, Class, Student.
1

1. Introduction

The modern era is the era of technology and science. The modern world is incredibly dynamic, and we are always being exposed to new technology advancements [1-2]. The influence of technology in every part of our life keeps expanding as the 21st century progresses. Education is one area where technology has had a significant influence [3]. All nations are following the global trend toward technology in order to create a competitive economy and raise the standard of living for their citizens. Technology plays a crucial role in the world we now live in. Modern education now includes a significant and revolutionary role for technology. As classrooms switch from static chalk-and-board settings to dynamic, tech-infused learning spaces, the educational landscape has recently undergone a seismic change [4]. The understanding that technology has the potential to improve accessibility, engagement, and effectiveness in education has been the driving force behind this transition [5].

The digital materials have broadened the scope of education and given students access to a variety of information at any time and from any location. With immersive and hands-on learning opportunities, artificial intelligence, virtual reality, and augmented reality are altering how students perceive and engage with educational information [6]. Communication is being improved through collaboration technologies and data analytics, which are also forming more individualized teaching plans.But as technology advances, it also brings with it problems like the digital divide and privacy issues. Though a paradigm change, the incorporation of technology into education promises to provide students the knowledge and adaptability they need to succeed in our increasingly digital and linked society. This extensive transition emphasizes how important it is for modern educational institutions to comprehend and utilize technology's possibilities**(Fig 1)**[7].

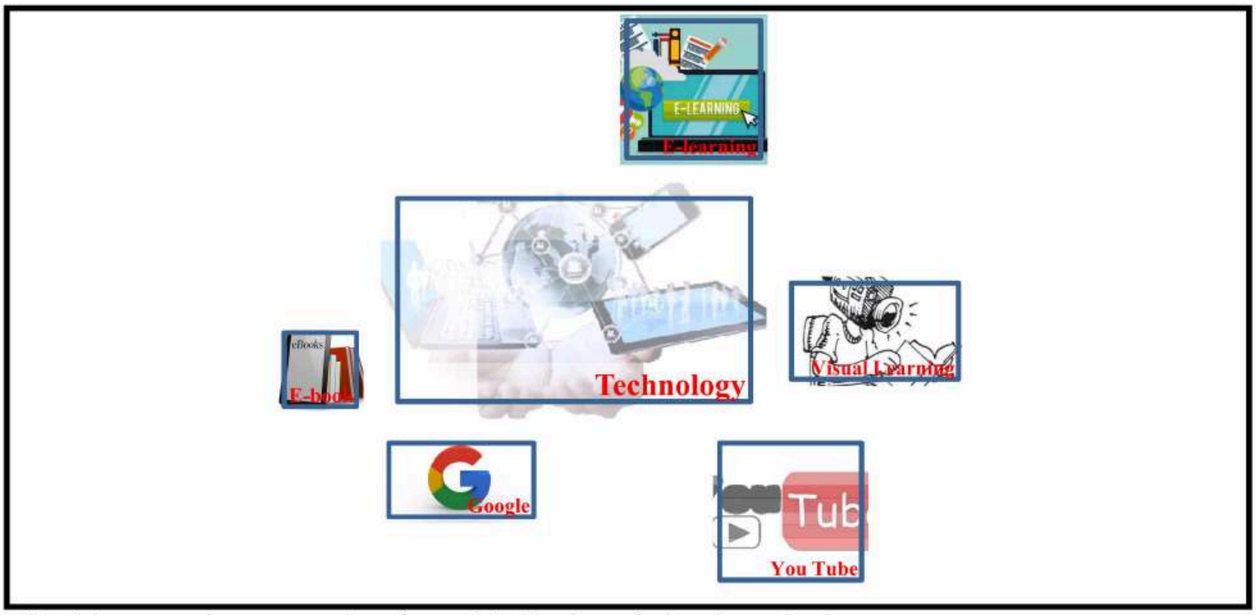


Fig. 1 A systematic representation of essential objectives of education technology.

2. Modern technology in education

Today's technology is a vital part of our lives. It is regarded as the cornerstone of economic expansion. In the current environment, a technology-deficient economy cannot expand. This is due to how much easier and faster technology has made our work [8]. Every potential subject is affected by technology, and education is one of them. Education has undergone a revolution because to modern technology, which has improved accessibility, engagement, and tailored instruction. By allowing remote learning, online platforms eliminate regional restrictions [9]. Virtual reality and interactive applications produce immersive learning experiences that boost student engagement. Individual learning demands may be met through customized content distribution made possible by big data analytics and AI-driven solutions. Students and instructors may communicate and share resources easily thanks to collaboration tools and cloud-based platforms. But there are issues that must be resolved, like the digital gap and privacy worries[10]. All things considered, current technology empowers teachers and students, enabling a flexible and dynamic educational environment that equips pupils for the challenges of the digital age.

The usage of current tools and equipment boosts students' learning and involvement, according to the most recent research on how exactly modern students choose to use technology today and how using technology affects their learning. When technology is used to help, they find it to be lot more engaging and fuller of intriguing places. Knowledge transfer becomes incredibly simple, practical, and efficient. This indicates that, in every area of life, including schooling, our minds now tend to function more quickly when supported by contemporary technology**(Fig. 2)**[11].

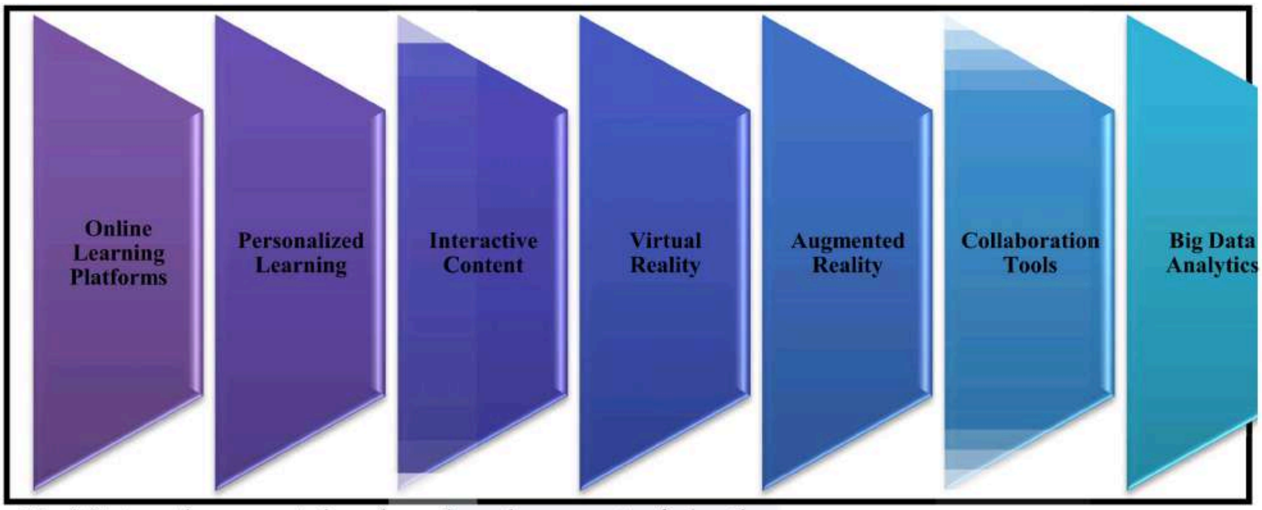


Fig. 2 Systematic representation of somekey advancements of education.

3. Why need of technology in modern education system

Digital technology must already be used in education due to their globalization. For teaching, resource sharing, evaluation, and administering the daily operations of academic institutions, online platforms were accessible. It will aid students in preparing for occupations that include using wireless technology in the future. In both teaching and learning, technology is quite helpful [12]. Numerous websites that are beneficial to both instructors and students are readily available online. Students are encouraged and motivated to learn via technology [13-14]. The instructor also incorporates technology tools into their presentations.Teachers can monitor student progress, pinpoint areas for development, and optimize their lesson plans thanks to big data analytics. Learning results and curriculum development are optimized by this data-driven approach.Knowledge of technology is essential in a world that is becoming more and more digital. Students who are exposed to technology in the classroom get the digital literacy and problem-solving skills required to succeed in the job market of the twenty-first century [15].The cost of educational materials can be decreased by using online resources and digital textbooks, increasing access to and affordability of education.Digital materials are environmentally benign since they lessen the demand for paper and the environmental impact of education**(Fig. 3)**[16].

Technology improves accessibility, engagement, and customization in the current educational system while preparing students for the needs of a digitally driven society. It promotes diversity, international cooperation, and data-driven development, eventually resulting in a more adaptable, efficient, and fair educational experience [17].

Fig. 3A systematic representation of digital classroom.

4. The role of technology in modern education system

Technology has a significant impact on today's educational system, changing how students learn and how professors impart knowledge. Here are a few main arguments in favour of technology in education [18].

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4.1. Access to information

Technology makes it simple to access large volumes of knowledge and learning materials. Students can use the internet to conduct research, access online classes, browse digital libraries, and learn things that aren't included in traditional textbooks.

4.2. Enhanced learning experience

Technology improves education by making it more individualized, interactive, and exciting. Simulators, virtual reality, educational software, and multimedia tools may all be used to bring abstract ideas to life and improve learning and memory [19-20].

4.3. Collaboration and communication

Collaboration and communication between students, instructors, and classmates are made possible through technology. Online platforms, forums, video conferencing, and collaboration tools make it simple to share ideas, work in teams, and build relationships around the world. Beyond the classroom, students may work together on projects, get feedback, and have important conversations [21].

4.4. Individualized learning

Learning experiences may be individualized and adaptive thanks to technology. Software and platforms for education may identify the requirements of each student and offer materials and lessons that are specifically suited to those needs. In order to promote unique learning routes, adaptive learning systems can modify the pace, degree of difficulty, and material based on each student's development and learning preferences [22].

4.5. Remote and online learning

As was evident during the COVID-19 epidemic, technology has proven especially important during periods of remote or distance schooling. Students may access educational resources and communicate with teachers remotely thanks to online learning platforms, video conferencing technologies, and educational applications [23].

4.6. Data-driven insights

The gathering and analysis of data on student performance, engagement, and advancement are made possible by technology. Learning management systems and educational analytics may offer instructors and administrators useful information that can be used to pinpoint problem areas, monitor student progress, and make data-driven decisions that will improve educational results [24].

4.7. Skill development

Students are given the necessary digital literacy abilities for the 21st century thanks to technology. Students gain knowledge about how to use digital tools, assess information critically, communicate clearly online, work remotely, and adjust to quickly changing technology advances. Success in the digital era requires certain abilities [25].

4.8. Accessibility and inclusivity

By making education accessible to students with a range of needs and skills, technology fosters inclusion. Students with impairments can engage completely in educational activities by using assistive technology like screen readers, captioning tools, and adapted equipment, which can remove learning obstacles [26].

In general, the use of technology in the current educational system empowers students, improves teaching techniques, increases access to knowledge, and gets pupils ready for a technologically advanced society. It encourages individualized instruction, group work, critical thinking, and provides students with the tools they need to succeed in the future [27-28].

5. Factersaffecting technology in modern education system

Technology performance is a complex process that depends on its uniqueness, the relationships between human resources, and educational environments. The following elements are noted as having an impact on educational technology use [29].

5.1.Access to inappropriate content

The main worry about technology use is how simple it is to get and see pornographic, violent, and other improper information.

5.2.Teacher's factor

The teacher is connected to a collection of elements that are frequently highlighted as influencing how technology is used in education. The primary aspect relating to the use of technology has always been recognised as the instructors' opinions on how to and competency with it. If a teacher doesn't use technology in the classroom, they should not hold to positive views about it. Additional elements that seem to encourage the correct use of technology in education include the instructive attitude and instructional practices of the teacher [30-31].

5.3.A disconnected youth

When individuals are glued to their screens virtually constantly, it has a negative impact on society and is resulting in a brand-new set of societal problems.

5.4.Technology factors

Technology itself is one of the many elements that influence how teachers use it. Contradictory ideas on the important effects of technology should be used in education today. This causes the instructors to become unsure of the appropriate educational ethics of technology. Teachers also find it challenging to keep up with the most recent technological developments due to the constantly evolving technologies. This is due to the fact that new gear and software are released every day, and instructors find it difficult and intimidating to keep up with this enigmatic beast of technology [32-33]. The following barriers are also frequently mentioned:

- ✓ Limited time
- ✓ Absence of access
- ✓ Inadequate resources
- ✓ Lack of knowledge
- ✓ Lack of assistance

When students use their mobile phones or other devices in class, their attention spans substantially decrease. Their teacher and lessons become less important as attention is diverted to what they are seeing, playing, or doing on their phones [34].

6. Challenges of technology in modern education system

The digital gap, which occurs when not all pupils have equal access to technology and causes educational differences, is one of the challenges of technology in contemporary education. Data gathering and storage for individualized learning raise privacy issues [35]. Effective technological integration is hampered by inadequate teacher preparation, and maintaining content quality despite the wealth of internet resources is difficult. Students' well-being and concentration are impacted by excessive screen time and technological distractions. Concerns that persist include providing fair access, resolving technological problems, and juggling innovation with conventional teaching techniques [36]. To protect student rights and uphold ethical standards in education, it is important to manage AI, data analytics, and surveillance issues carefully.Technology has significant challenges, particularly in its implementation and use. Concerns about excessive screen time, the effectiveness of instructors' use of technology, and technological equity issues are also brought up. The COVID-19 issue has increased the importance of the material [37].

In order to guarantee that technology's advantages are widely available, equally distributed, and morally upstanding, it is crucial to overcome these difficulties. Successfully navigating the changing world of technology in the current educational system requires a careful, balanced approach, continual study, and adaptation **(Fig. 4)**[38]**.**

Fig. 4 Systematic representation of challenges of technology in modern education.

7. Future prospects for technology in modern education

Technology in education is primed for transformational expansion in the future. AI-powered personalized learning will spread, adjusting schooling to individual requirements. Immersive, interactive experiences will be possible with virtual and augmented reality [39]. Expanding online and mixed learning will give more people flexible access to high-quality education. Data analytics will improve learning outcomes by improving teaching strategies. New technology will promote digital literacy and critical thinking abilities. Global collaboration will thrive, fostering a variety of viewpoints [40]. It will become more accessible to everyone, including those with impairments.All linguistic barriers will vanish, and there will be more regional language learning materials available online. Programs for m-learning and e-learning give teachers and students access to a huge library of informational resources [41]. While technology will play a crucial part in determining the future of education, effective use of new teaching tools will depend on a new generation of teachers who recognize the value of interpersonal interaction in the classroom. However, for a successful future in education, ethical issues like data privacy and AI ethics will need to be carefully considered and regulated**(Fig. 5)**[42].

In conclusion, the use of technology in education has enormous promise for innovation, customization, and accessibility. To fully capitalize on these opportunities and build a more effective and fair educational system, it will be necessary to address issues relating to equality, privacy, and ethics [43].

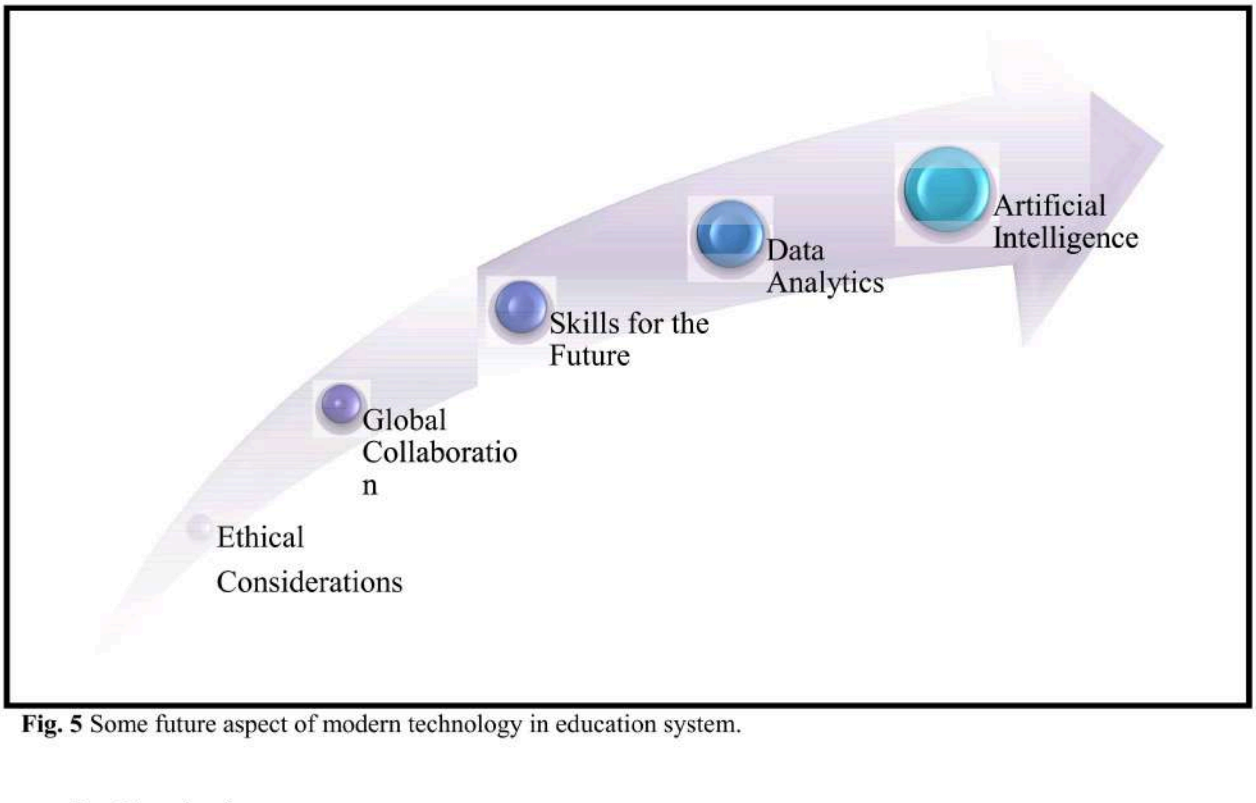

Fig. 5 Some future aspect of modern technology in education system.

8. Conclusion

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The chapter "Need and Application of Technology in Modern Education System" concludes by highlighting the unquestionable importance of technology in redefining modern education. It has become clearly evident that technology is essential in today's educational environment and not just a choice. The use of technology in education has changed how we teach and learn. Examples include e-learning platforms, virtual classrooms, educational apps, and online assessment tools. In order to accommodate the various demands and learning preferences of students, it provides accessibility, flexibility, and scalability. Additionally, technology has the ability to reduce educational disparities and provide students throughout the world access to high-quality instruction.The digital divide and worries about too much screen time are only two of the difficulties that come with using technology in education, so we must always be aware of them. To maximize technology's beneficial effects, these issues must be addressed and the technology must be used appropriately. In summary, technology is a necessary tool for modern education because it allows us to design learning environments that are more effective, inclusive, and engaging. Utilizing technology's potential while juggling creativity and responsibility will be essential as we traverse the digital era in making sure that education continues to change and adapt to suit the ever-changing requirements of students and instructors.

Authors contribution

The manuscript was written through the contributions of both authors. All authors have approved the final version of the manuscript.

Notes

All authors declare no competing financial interest.

Conflicts of interest

There are no conflicts to declare.

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Research Paper

CONVERGENCE ANALYSIS OF PROPORTIONAL-DERIVATIVE -TYPE ILC FOR LINEAR CONTINUOUS CONSTANT TIME DELAY SWITCHED SYSTEMS WITH OBSERVATION NOISE AND STATE UNCERTAINTIES

Omprakash Dewangan

Abstract. This article is concerned with the linear continuous time delay switching system with state uncertainties and observation noise. The goal of this study is to investigate how an internal switching mechanism and the efficacy of a conventional proportional-derivative ILC method is impacted by ambient noise for linear continuous-time switching systems. The findings demonstrate that learning gains and the dynamics of the subsystems, rather than the time-driven switching rule, are primarily responsible for the convergence and robustness of the control method. An appropriate selection of learning gains can ensure the control algorithm's convergence and resilience given any arbitrary time-varying switching rule.

Key Words: Iterative learning control, switched system, dynamical system, time delay, bounded state disturbance, bounded observation noise.
2010 Mathematics Subject Classification: 70G10, 39B12, 39A10

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1. Introduction

Theoretical research and practical applications of switched systems, which contain a given finite number of subsystems and switching signals, have recently received a great deal of interest. When a link in a network fails or is created, the connection topology frequently changes. Because the reference trajectory is established over a finite period, an ILC system repeatedly performs the same finite duration operation. The duration is referred to as the pass length, and each repetition is referred to as a pass. The system is brought back to its starting point when each pass is finished, so that the next pass may begin. The systems might diverge as a result of the states being reset, which could result in positioning problems. Through repeated completion of the same tasks, an ILC law that combines the knowledge from past passes with that from the current pass can eventually bring the output to the reference trajectory. Many ILC laws, including PID-type, P-type, and D-type ILC, PD-type, have been suggested for various kinds of systems. For instance, a hard disc drive's track following duty, a wafer manufacturing process's temperature management task, etc. When we refer to an extensive system, it means one that is made up of several subsystems that are connected by the system's extensive state vector, but each of which is managed based on its own input and output data. Examples of typical large scale systems include petrochemical operations, electricity systems, networked control systems, etc. According to Chen at. al. [1], a system for learning at the beginning that operates between two successive iterations, establishes the starting position at a certain location, and asymptotically converges is suggested.

In 1993, according to Hwangat. al. [2] the Derivative type ILC is built for reliable continuous-time systems, which are linear, and by this, we mean that the systems are fed a comparison of tracking error. One of the key issues that occurs with switched systems is stability, which has drawn the most attention. To analyze the stability of switched systems, a variety of techniques have been developed, and numerous helpful stability criteria have been defined in some articles. In order to ensure system stability and improve system performance, the dwell-time approach has been successful in determining the proper switching signals for switched systems that are subject to controlled switching signals

Ruan at. al. [4] offer a PID type control update method that follows non-repeated goal trajectories. The technique is demonstrated to be limited in the Lp norm sense. It is well knowledge that many engineering systems inevitably have time delays. It is possible for the system to become unstable if the time delays are not properly managed. A type of time delay system known as a neutral system depends not only on state delay but also on state derivative delay. In [3], the requirements for switching delay systems' delay-dependent exponential stability are provided [5]-[10]. Due to its hybrid nature, a switched system typically does not inherit subsystem characteristics [12].

In some cases, alternating between these reliable sub-systems may even cause the switched system to become unstable. For instance, the stability which is global exponential, trait of all subsystems cannot ensure the switched system has the same stability attribute. Therefore, switched systems are not immediately applicable to typical design and analysis techniques for systems without switching. Evidently, switched systems are rife with uncertainty, which complicates the research of switched systems even further. It is anticipated that adaptive control, which is an effective method for researching ambiguous non-switched systems, will also be useful for research of switched systems with uncertainty [13]-[16]. In reality, this presumption is frequently unfounded [9].

For a class of LCTSSs, which may be recognized by random time-driven switching signals and observation noise interference, the learning performance of a classic PD-type ILC scheme was examined by Xaun Yang et al. in 2018 [18]. A necessary condition of convergence and robustness is derived by incorporating using some lemma, and the impact of switching and noise is examined.

The rest of this essay is structured as follows. Preliminary, concept property and lemma related information are given in Section 2. The tracking effectiveness of a class of linear continuous time delay switched systems with observation noise and state uncertainties using PD- type ILC is examined in section 3. The paper is wrapped up in the final part.

2. **Basic and Mathematical Formulation**

Take into consideration a class of linear continuous time delay switching systems with state uncertainties:

$$\begin{cases} \dot{x}_k(t) & = A_{\sigma(t)}x_k(t) + D_{\sigma(t)}x_k(t-\tau) + B_{\sigma(t)}u_k(t) + \xi_k(t), \\ y_k(t) & = C_{\sigma(t)}x_k(t) + w_{\sigma,k}(t), \quad t \in \Omega = [0,T], \end{cases} \tag{2.1}$$

here

(1) $k \in \mathbb{N}$ represent the number of iterations, $\Omega = [0, T]$ denotes the time interval and $t \in \Omega$ denotes variable for time, τ denotes delay in time;
(2) $x_k(t)$ be element $\mathbb{R}^n$,, which stands for state vector, $u_k(t)$ is an element in $\in \mathbb{R}^m$, which stands for input vector and $y_k(t)$ is element in $\mathbb{R}^l$ is output vector.
(3) $w_{\sigma,k}(t) \in \mathbb{R}^l$ and $\xi_k(t)$ denotes the bounded state disturbance bounded observation noise with $\|w_{\sigma(i),k}\|_p \leq w_{i,0}$ and $\|\xi_k(t)\|_p \leq b_\xi$;
(4) $A_\sigma(t)$ be the matrix in $\in \mathbb{R}^{n\times n}$, $B_{\sigma(t)}$ be the matrix in $\mathbb{R}^{n\times m}$ and $C_{\sigma(t)}$ are also matrix in $\mathbb{R}^{l\times n}$, this all type of matirx are known as system matrices;
(5) $\sigma : [0, T] \to Q, \quad Q = \{1, 2, \cdots, q\}$ over a period of time, $[0, T]$ denotes a piecewise constant function which is known as the switching rule.

Without harming generality, it is thought to be characterized as

$$\sigma(t) = i = \begin{cases} 1, & t \text{ belong to } [0, t_1), \\ 2, & t \text{ belong to } [t_1, t_2), \\ \vdots & \\ q, & t \text{ belong to } [t_{q-1}, T]. \end{cases} \tag{2.2}$$

Therefore, the matrices group $(A_{\sigma(t)}, B_{\sigma(t)}, C_{\sigma(t)}, D_{\sigma(t)})$, for $\sigma(t)$ belong to $Q = \{1, 2, \cdots, q\}$ are a component of the ensuing the following set

$$\{(A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), \cdots, (A_q, B_q, C_q, D_q)\}$$

Satisfied (2.2), the system (2.1) is perhaps reformed as

$$\begin{cases} \dot{x}_k(t) & = A_i x_k(t) + B_i u_k(t) + D_i x_k(t-\tau) + \xi_k(t), \\ y_k(t) & = C_i x_k(t) + w_{i,k}(t), \quad t \text{ belong to } \Omega = [0, T], i \in Q. \end{cases} \tag{2.3}$$

Keep in mind that the dynamic system (2.3) can function repeatedly across the range $[0, T]$ of time, which is finite, even if the precise dynamics may not be known.

Consider the scheme, which is known as PD- type ILC as follows:

$$u_{k+1}(t) = u_k(t) + \Gamma_{p,i} e_k(t) + \Gamma_{d,i} \dot{e}_k(t), \quad i \in Q = \{1, 2, \cdot, q\}, k = 1, 2, 3, ... \tag{2.4}$$

is imposed the k^{th} term of error, which is denoted by $e_k(t)$ and define as $e_k(t) = y_d(t) - y_k(t)$, for any t belong to finite time interval $[0, T]$ is known as the tracking error, and $\Gamma_{d,i} \in \mathbb{R}^{m\times l}$ and $\Gamma_{p,i} \in \mathbb{R}^{m\times l}$ are known as derivative gains and proportional gains, respectively. The purpose is that the output of the system (2.3) asymptotically converges to the given targeted or reference trajectory, which is indicated by $y_d(t)$, in time period $t \in [0, T]$ as exactly as feasible or when the iteration number tence to infinity follows into the vicinity of $y_d(t)$, that is,

$$\lim_{k\to\infty} \|e_{k+1}(\cdot)\|_p = 0 \text{ or } \lim_{k\to\infty} \sup \|e_{k+1}(\cdot)\|_p \leq \eta.$$

To object of this problem, to find the sequence $\{u_k(t) : k \in \mathbb{Z}_+\}$ such that $\{y_k(t)\}$ tends to $y_d(t)$ for (2.1) with PD-type ILC scheme(2.2).

Definition 2.1. [18] Consider the vector valued function $g : I \subseteq \mathbb{R}^+ \to \mathbb{R}^n$ defined by

$$g(t) = [g_1(t), g_2(t), \cdots, g_n(t)]^T,$$

its Lebesgue -p norm is defined as

$$\|g(\cdot)\|_p = \left[\int_I \left(\max_{1\leq j\leq n}\{|g_j(t)|\}\right)^p dy\right]^{\frac{1}{p}}, 1 \leq p \leq \infty.$$

Definition 2.2. [19]

For a given vector valued function $f(t) \in \mathbb{R}^n, g(t) \in \mathbb{R}^n$, the convolution integral is described as

$$(f * g)(t) = \int_I f(t-s)g(s)ds.$$

From definition (2.1) and (2.2), The convolution integrals generalized Young inequality (GYI) is stated as

$$\|f(\cdot)\|_q \|g(\cdot)\|_p \geq \|(f * g)(\cdot)\|_r, \tag{2.5}$$

for all $1 \leq p, q, r < \infty$ satisfying

$$1/r = 1/p + 1/q.$$

In particular, if p and r are equal, then inequality (2.5) , we get

$$\|f\|_1 \|g\|_p \geq \|f * g\|_p, \tag{2.6}$$

when $p = r$.

The following are the system's (2.3) basic presumptions:

Assumption 1: Every operation begins at the same starting place. In this paper, it's thought to be so $y_d(0) = y_k(0)$, for all $k = 1, 2, \cdots$.

Assumption 2: The given targeted or reference or desired output $y_d(t)$ is invariant in the process of iteration over a time interval $[0, T]$.

Assumption 3: The switched sequence $\sigma(t)$ still maintains iteration invariance and at the first iteration, it is randomly chosen.

Assumption 4: For every $t \in [0, T]$, there is $\Delta x_k(-t) = 0$.

Assumption 5: Over every time sub-interval $[t_{i-1}, t_i], i \in Q,$, the observation noise is arbitrarily constrained, it means, $w_{i,k}(t) \leq w_{i,0}$ where, each time subinterval's value of $w_{i,0}$ is to little enough non-negative constant.

Assumption 6: The state uncertainty (disturbance) is bounded, that is, $\|\xi_k\|_p \leq b_\xi$.

Assumption 7: Regarding the specified intended result
$y_d(t)$, the only thing present is a desired control input $u_d(t)$ and a desired $x_d(t)$ s. t.

$$\begin{cases} \dot{x}_d(t) & = A_i x_d(t) + D_i x_d(t-\tau) + B_i u_d(t) + \xi_d(t), \\ y_d(t) & = C_i x_d(t), \quad t \in \Omega = [0, T], i \in Q. \end{cases}$$

Here τ denotes the time delay so that the dwell times of every subsystem exceed the delay times. That is,

$$\tau < t_i - t_{i-1}, \quad \forall i \in Q.$$

3. Main Results

Lemma 3.1. [19] *Assume that $\{b_k\}$ is a positive sequence of a real sequence defined as follows:*

$$b_k \leq c_1 b_{k-1} + c_2 b_{k-2} + \cdots + c_n b_{k-n} + \varepsilon_k, \quad k = n+1, n+2, \cdots,$$

with the starting value b_l for every $l = 1, 2, \cdots, n$, where $\{\varepsilon_k\}$ is another specified real sequence. If the coefficient satisfy $c_j \geq 0$ and

$$c = \sum_{j=1}^{n} c_j < 1,$$

then the $\limsup_{k\to\infty} \varepsilon_k \leq \varepsilon$ *implies that*

$$\limsup_{k\to\infty} b_k \leq \frac{\varepsilon}{1-c}.$$

In particular, $\lim_{k\to\infty} b_k = 0$, *provided that* $\varepsilon = 0$.

Theorem 3.2. *Consider the scheme (2.4) that is imposed on the system (2.3), which is defined by the switching rule (2.4) and is affected by uncertainties and noise. Assume that the system (2.3) satisfies assumes from A1 to A7. If the* A_i, B_i, C_i *and* D_i *are system dynamics together with the learning gains* $\Gamma_{d,i}$ *and* $\Gamma_{p,i}$ *satisfy*

$$\|C_i \exp(A_i\cdot(\cdot))(A_iB_i\Gamma_{d,i} + B_i\Gamma_{p,i})\|_1 + \|I - C_iB_i\Gamma_{d,i}\|_\infty = \rho_i < 1, \tag{3.1}$$

for every sub-system, then the system output $y_k(t)$ *can approach the neighborhood of the targeted trajectory* $y_d(t)$ *in the whole time interval, as the iteration num tends to infinity.*

Proof. Firstly, consider the input control signal $u_k(t)$ in the k^{th} trial over time sub interval $[t_{i-1}, t_i](i \in Q)$, the state response trajectory of the system (2.3) is formally represented as

$$\begin{aligned} x_{k+1}(t) = &\exp(A_i\cdot(t-t_{i-1}))x_{k+1}(t_{i-1}) \\ &+ \int_{t_{i-1}}^{t} \exp(A_i\cdot(t-s))D_ix_{k+1}(s-\tau)(s)ds \\ &+ \int_{t_{i-1}}^{t} \exp(A_i\cdot(t-s))B_iu_{k+1}(s)ds \\ &+ \int_{t_{i-1}}^{t} \exp(A_i\cdot(t-s))\xi_{k+1}(s)ds. \end{aligned}$$

Using the recursive relationship of tracking errors, the tracking error $e_{k+1}(t)$ is therefore described as follows:

$$\begin{aligned}
e_{k+1}(t) &= y_d(t) - y_{k+1}(t) \\
&= y_d(t) - y_k(t) - [y_{k+1}(t) - y_k(t)] \\
&= e_k(t) - C_i \exp(A_i \cdot (t - t_{i-1}))(x_{k+1}(t_{i-1}) - x_k(t_{i-1})) \\
&- C_i \int_{t_i}^{t} \exp(A_i \cdot (t-s)) D_i [x_{k+1}(s-\tau) - x_k(s-\tau) ds \\
&- C_i \int_{t_{i-1}}^{t} \exp(A_i \cdot (t-s)) B_i [u_{k+1}(s) - u_k(s)] ds \\
&- C_i \int_{t_{i-1}}^{t} \exp(A_i \cdot (t-s)) [\xi_{k+1}(s) - \xi_k(s) ds - (w_{i,k+1}(t) - w_{i,k}).
\end{aligned} \tag{3.2}$$

Now, we consider the PD type ILC as an updating law (2.4), which is a substitute in the above equation (3.2), we can easily calculate as follows:

$$\begin{aligned}
e_{k+1}(t) &= e_k(t) - C_i \exp(A_i \cdot (t - t_{i-1}))(x_{k+1}(t_{i-1}) - x_k(t_{i-1})) \\
&- C_i \int_{t_i}^{t} \exp(A_i \cdot (t-s)) D_i [x_{k+1}(s-\tau) - x_k(s-\tau) ds \\
&- C_i \int_{t_{i-1}}^{t} \exp(A_i \cdot (t-s)) B_i [\Gamma_{p,i} e_k(s) + \Gamma_{d,i} \dot{e}_k(s)] ds \\
&- C_i \int_{t_{i-1}}^{t} \exp(A_i \cdot (t-s)) [\xi_{k+1}(s) - \xi_k(s) ds - (w_{i,k+1}(t) - w_{i,k}) \\
&= e_k(t) - C_i \exp(A_i \cdot (t - t_{i-1}))(x_{k+1}(t_{i-1}) - x_k(t_{i-1})) \\
&- C_i \int_{t_i}^{t} \exp(A_i \cdot (t-s)) D_i [x_{k+1}(s-\tau) - x_k(s-\tau) ds \\
&- C_i \int_{t_{i-1}}^{t} \exp(A_i \cdot (t-s)) B_i \Gamma_{p,i} e_k(s) ds \\
&- C_i \int_{t_{i-1}}^{t} \exp(A_i \cdot (t-s)) [\xi_{k+1}(s) - \xi_k(s) ds \\
&- C_i \int_{t_{i-1}}^{t} \exp(A_i \cdot (t-s)) B_i \Gamma_{d,i} \dot{e}_k(s) ds - (w_{i,k+1}(t) - w_{i,k}).
\end{aligned} \tag{3.3}$$

By using the partial integration approach, it is possible to rearrange the last term in equation (3.3) to become as follows:

$$\begin{aligned} C_i \int_{t_{i-1}}^{t} \exp(A_i \cdot (t-s)) B_i \Gamma_{d,i} \dot{e}_k(s) ds \\ &= C_i \exp(A_i \cdot (t-s)) B_i \Gamma_{d,i} e_k(s) \big|_{s=t_{i-1}}^{s=t} \\ &- C_i \int_{t_{i-1}}^{t} \exp(A_i \cdot (t-s))(A_i B_i \Gamma_{d,i} \\ &+ B_i \Gamma_{p,i}) e_k(s) ds. \end{aligned} \tag{3.4}$$

Substituing (3.4) into (3.3) yields

$$\begin{aligned} e_{k+1}(t) = e_k(t) &- C_i \exp(A_i \cdot (t - t_{i-1}))(x_{k+1}(t_{i-1}) - x_k(t_{i-1})) \\ &- C_i \int_{t_i}^{t} \exp(A_i \cdot (t-s)) D_i [x_{k+1}(s-\tau) - x_k(s-\tau) ds \\ &- C_i \int_{t_{i-1}}^{t} \exp(A_i \cdot (t-s)) B_i \Gamma_{p,i} e_k(s) ds \\ &- C_i \int_{t_{i-1}}^{t} \exp(A_i \cdot (t-s)) [\xi_{k+1}(s) - \xi_k(s) ds \\ &- C_i \exp(A_i \cdot (t-s)) B_i \Gamma_{d,i} e_k(s) \big|_{s=t_{i-1}}^{s=t} \\ &- C_i \int_{t_{i-1}}^{t} \exp(A_i \cdot (t-s))(A_i B_i \Gamma_{d,i} + B_i \Gamma_{p,i}) e_k(s) ds \\ &- (w_{i,k+1}(t) - w_{i,k}). \end{aligned} \tag{3.5}$$

Step 1: If t belongs to the first sub-interval $t \in \Omega_1$. The first subsystem is turned on in this situation. Taking $t_0 = 0$, the tracking error's recursive connection (3.4) becomes

$$\begin{aligned} e_{k+1}(t) &= e_k(t) - C_i \exp(A_1 \cdot (t))(x_{k+1}(0) - x_k(0)) \\ &\quad - C_1 \int_0^t \exp(A_1 \cdot (t-s)) D_1 [x_{k+1}(s-\tau) - x_k(s-\tau)] ds \\ &\quad - C_1 \int_0^t \exp(A_1 \cdot (t-s)) B_1 \Gamma_{p,i} e_k(s) ds \\ &\quad - C_1 \int_0^t \exp(A_1 \cdot (t-s)) [\xi_{k+1}(s) - \xi_k(s)] ds \\ &\quad - C_1 \exp(A_1 \cdot (t-s)) B_1 \Gamma_{d,1} e_k(s) \Big|_{s=t_{i-1}}^{s=t} \\ &\quad - C_1 \int_0^t \exp(A_1 \cdot (t-s))(A_1 B_1 \Gamma_{d,1} \\ &\quad + B_1 \Gamma_{p,1}) e_k(s) ds. - (w_{1,k+1}(t) - w_{1,k}). \end{aligned} \tag{3.6}$$

Using first assumption A1, which is $(x_{k+1}(0) - x_k(0)) = 0$. Thus, we have

$$\begin{aligned} e_{k+1}(t) &= (I - C_1 B_1 \Gamma_{d,1}) e_k(t) \\ &\quad - C_1 \int_0^t \exp(A_1 \cdot (t-s))(A_1 B_1 \Gamma_{d,1} + B_1 \Gamma_{p,1}) e_k(s) ds \\ &\quad - C_1 \int_0^t \exp(A_1 \cdot (t-s)) D_1 [x_{k+1}(s-\tau) - x_k(s-\tau)] ds \\ &\quad - C_1 \int_0^t \exp(A_1 \cdot (t-s)) [\xi_{k+1}(s) - \xi_k(s)] ds \\ &\quad - (w_{1,k+1}(t) - w_{1,k}). \end{aligned} \tag{3.7}$$

Since $\Delta w_{1,k}(t) = w_{1,k+1}(t) - w_{1,k}$, then we get as follows:

$$\begin{aligned} e_{k+1}(t) &= (I - C_1 B_1 \Gamma_{d,i}) e_k(t) - C_1 \int_0^t \exp(A_1 (t-s))(A_1 \cdot B_1 \Gamma_{d,1} \\ &\quad + B_1 \Gamma_{p,1}) e_k(s) ds - C_1 \int_0^t \exp(A_1 \cdot (t-s)) D_1 \Delta x_k(s-\tau) ds \\ &\quad - C_1 \int_{t_0}^t \exp(A_1 \cdot (t-s)) \Delta \xi_k(s) ds - \Delta w_{1,k}(t). \end{aligned} \tag{3.8}$$

where $\Delta x_k(s-\tau) = x_{k+1}(s-\tau) - x_k(s-\tau)$, $\Delta \xi_k(s) = \xi_{k+1}(s) - \xi_k(s)$ and $\Delta w_{1,k} = w_{1,k+1}(t) - w_{1,k}(t)$. Firstly, applying the Lebesgue-p norm on two sides of the equation (3.8) and using the definitions (2.1) and (2.2), we get

$$
\begin{aligned}
\|e_{k+1}(\cdot)\|_p \leq & (\|(I - C_1 B_1 \Gamma_{d,1})\|_\infty + \|C_1 \exp(A_1(t-s))(A_1 \cdot B_1 \Gamma_{d,i} \\
& + B_1 \Gamma_{p,1})\|_1) e_k(\cdot)\|_p + \|C_1 \exp(A_1 \cdot (t-s)) D_1\|_p \|\Delta x_k(s-\tau)\|_p \\
& + \|\exp(A_1(t-s)\|_p \|\Delta\xi_k(t)\|_p + \|\Delta w_{1,k}(t)\|_p \\
& \leq \|(I - C_1 B_1 \Gamma_{d,1})\|_\infty \|e_k(\cdot)\|_p + \|C_1 \exp(A_1 \cdot (t-s)) \\
& (A_1 B_1 \Gamma_{d,1} + B_1 \Gamma_{p,1})\|_1 \|e_k(\cdot)\|_p \\
& + \|C_1 \exp(A_1(t-s)) D_1\|_p \gamma_0 + \|C_1 \exp(A_1 \cdot (t-s)\|_p b_\xi + 2b_{w_{1,0}} \qquad (3.9)
\end{aligned}
$$

Where $\|\Delta x_k(s-\tau)\|_p \leq \gamma_0$, and we can observe that

$$
\begin{aligned}
\|\Delta\xi_k(t)\|_p \leq & \|\xi_{k+1}(t)\|_p + \|\xi_k(t)\|_p \leq 2b_\xi, \\
\|\Delta w_{1,k}(t)\|_p \leq & \|w_{1,k+1}(t)\|_p + \|w_{1,k}(t)\|_p \leq 2w_{1,0},
\end{aligned}
$$

So, $\|\Delta\xi_k(t)\|_p$ and $\|\Delta w_{1,k}(t)\|_p$ are bounded by $2b_\xi$ and $2w_{1,0}$ respectively. Taking the supremum of the equation (3.9) with the assumption $\rho_1 < 1$ and applying the Lemma 3.1, we conclude that

$$
\begin{aligned}
\lim_{k\to\infty} \|e_{k+1}(\cdot)\|_p \leq & \frac{\|C_1 \exp(A_1 \cdot (t-s)) D_1\|_p \gamma_0}{1-\rho_1} \\
& \frac{+\|C_1 \exp(A_1 \cdot (t-s)\|_p b_\xi + 2b_{w_{1,0}}}{1-\rho_1} = \frac{\delta_1}{1-\rho_1}, \qquad (3.10)
\end{aligned}
$$

over $[0, t_1)$, where $\delta_1 = \|C_1 \exp(A_1 \cdot (t-s)) D_1\|_p \gamma_0 + \|C_1 \exp(A_1 \cdot (t - s)\|_p b_\xi + 2b_{w_{1,0}}$. In other words, the 1^{st} sub-system's output can follow the targeted trajectory towards a neighborhood on $\Omega_1 = [0, t_1)$.

Step 2: In the second step, t belongs to the second sub-system, $[t_1, t_2)$, The second subsystem is switched on in this situation. The tracking error (3.6) is expressed as follows:

$$
\begin{aligned}
e_{k+1}(t) = {} & e_k(t) - C_2 \exp(A_2 \cdot (t-t_1))(x_{k+1}(t_1) - x_k(t_1)) \\
& - C_2 \int_{t_1}^{t} \exp(A_2 \cdot (t-s)) D_2 [x_{k+1}(s-\tau) - x_k(s-\tau)] ds \\
& - C_2 \int_{t_1}^{t} \exp(A_2 \cdot (t-s)) [\xi_{k+1}(s) - \xi_k(s)] ds \\
& - C_2 \exp(A_2 \cdot (t-s)) B_2 \Gamma_{d,2} e_k(s) \big|_{s=t_1}^{s=t} \\
& - C_2 \int_{t_1}^{t} \exp(A_2 \cdot (t-s)) (A_2 B_2 \Gamma_{d,2} + B_2 \Gamma_{p,2}) e_k(s) ds \\
& - (w_{2,k+1}(t) - w_{2,k}) \\
= {} & (I - C_2 B_2 \Gamma_{d,i}) e_k(t) \\
& - C_2 \int_{t_1}^{t} \exp(A_2 \cdot (t-s)) (A_2 B_2 \Gamma_{d,2} + B_2 \Gamma_{p,2}) e_k(s) ds \\
& - C_2 \exp(A_2 \cdot (t-t_1)) \Delta x_k(t_1) + C_2 \exp(A_2 \cdot (t-t_1)) B_2 \Gamma_{d,2} e_k(t_1) \\
& - C_2 \int_{t_1}^{t} \exp(A_2 \cdot (t-s)) D_2 \Delta x_k(s-\tau) ds \\
& - C_2 \int_{t_1}^{t} \exp(A_2 \cdot (t-s)) \Delta \xi_k(s) ds + \Delta w_{2,k}(t).
\end{aligned}
\tag{3.11}
$$

Where $\Delta x(t_1)$ is equal to $x_{k+1}(t_1) - x_k(t_1)$, $\Delta x_k(s-\tau) = x_{k+1}(s-\tau) - x_k(s-\tau)$, $\Delta \xi_k(s) = \xi_{k+1}(s) - \xi_k(s)$ and $\Delta w_{2,k} = w_{2,k+1}(t) - w_{2,k}(t)$.

Using the generalized Young inequality of the convolution integral and the taking Lebesgue -p norm on both sides of the equation (3.11), and applying the definition (2.1) and (2.2), we can formulate as

$$\begin{aligned}
&\|e_{k+1}(\cdot)\|_p \\
&\leq (\|I - C_2B_2\Gamma_{d,2}\|_\infty + \|C_2\exp(A_2\cdot(\cdot))(A_2B_2\Gamma_{d,2} + B_2\Gamma_{p,2})\|_1)\|e_k(\cdot)\|_p \\
&+\|C_2\exp(A_2\cdot(t-t_1))\|_p\|\Delta x_k(t_1)\|_p + \|C_2\exp(A_2\cdot(\cdot))D_2\|_p\|\Delta x_k(s-\tau)\|_p \\
&+\|C_2\exp(A_2\cdot(\cdot))\|_p\|\Delta\xi_k(t)\|_p + \|C_2\exp(A_2\cdot(\cdot))B_2\Gamma_{d,2}\|_p\|e_k(t_1)\|_p \\
&\qquad +\|\Delta w_{2,k}(t)\|_p
\end{aligned}$$

$$\begin{aligned}
&\leq (\|(I - C_2B_2\Gamma_{d,2})\|_\infty + \|C_2\exp(A_2\cdot(\cdot))(A_2B_2\Gamma_{d,2} + B_2\Gamma_{p,2})\|_1)\|e_k(\cdot)\|_p \\
&+\|C_2\exp(A_2\cdot(t-t_1))\|_p\|\Delta x_k(t_1)\|_p + \|C_2\exp(A_2\cdot(\cdot))D_2\|_p\gamma_1 \\
&+\|C_2\exp(A_2\cdot(\cdot))B_2\Gamma_{d,2}\|_p\|e_k(t_1)\|_p + \|C_2\exp(A_2\cdot(\cdot))\|_pb_\xi + 2b_{w_{2,0}} \\
&= \rho_2\|e_k(\cdot)\|_p + \|C_2\exp(A_2\cdot(t-t_1))\|_p\|\Delta x_k(t_1)\|_p \\
&\qquad +\|C_2\exp(A_2\cdot(t-s))D_2\|_p\gamma_1 \\
&+\|C_2\exp(A_2\cdot(\cdot))B_2\Gamma_{d,2}\|_p\|e_k(t_1)\|_p + \|C_2\exp(A_2\cdot(t-s))\|_pb_\xi + 2b_{w_{2,0}}.
\end{aligned} \tag{3.12}$$

where $\|\Delta x_k(s-\tau)\|_p \leq \gamma_1, \|\Delta\xi_k\|_p \leq 2b_\xi, \|\Delta w_{2,k}\|_p \leq 2b_{w_{2,0}}$. It is seen that the proving procedure starts with the first sub-interval Ω_1 that

$$\begin{aligned}
&\lim_{k\to\infty}\|e_{k+1}(\cdot)\|_p \\
&\leq \frac{\|C_1\exp(A_1\cdot(t-s))D_1\|_p\gamma_0 + \|C_1\exp(A_1\cdot(t-s))\|_pb_\xi + 2b_{w_{1,0}}}{1-\rho_1} \\
&= \frac{\delta_1}{1-\rho_1},
\end{aligned}$$

Where $\delta_1 = \|C_1\exp(A_1\cdot(t-s))D_1\|_p\gamma_0 + \|C_1\exp(A_1\cdot(t-s))\|_pb_\xi + 2b_{w_{1,0}}$ satiesfy on first subinterval Ω_1, which implies both $\lim_{k\to\infty}\sup\|\Delta x_k(t_1)\|_p < \infty$ and $\lim_{k\to\infty}\sup\|e_k(t_1)\|_p < \infty$ are satiesfied. Now indecating $\lim_{k\to\infty}\sup\|\Delta x_k(t_1)\|_p = \alpha_1$ and $\lim_{k\to\infty}\sup\|e_k(t_1)\|_p = \beta_1$, the inequality (3.12) can be written as follows:

$$\begin{aligned}
\lim_{k\to\infty}\sup\|e_{k+1}(\cdot)\|_p \leq \rho_2\|e_k(\cdot)\|_p &+ \|C_2\exp(A_2\cdot(\cdot))\|_p\alpha_1 \\
&+\|C_2\exp(A_2\cdot(\cdot))D_2\|_p\gamma_1 \\
&+\|C_2\exp(A_2\cdot(\cdot))B_2\Gamma_{d,2}\|_p\beta_1 \\
&+\|C_2\exp(A_2\cdot(\cdot))\|_pb_\xi + 2b_{w_{2,0}}.
\end{aligned} \tag{3.13}$$

Again, applying Lemma 3.1, it follows that

$$\lim_{k\to\infty}\sup\|e_{k+1}(\cdot)\|_p \leq \frac{1}{\rho_2}\|C_2\exp(A_2\cdot(\cdot))\|_p\alpha_1+\|C_2\exp(A_2\cdot(\cdot))D_2\|_p\gamma_1 +\|C_2\exp(A_2\cdot(\cdot))B_2\Gamma_{d,2}\|_p\beta_1+\|C_2\exp(A_2\cdot(\cdot))\|_p b_\xi + 2b_{w_{2,0}}$$

$$= \frac{\delta_2}{1-\rho_2}. \tag{3.14}$$

where $\delta_2 = \|C_2\exp(A_2\cdot(\cdot))\|_p\alpha_1+\|C_2\exp(A_2\cdot(\cdot))D_2\|_p\gamma_1+\|C_2\exp(A_2\cdot(\cdot))B_2\Gamma_{d,2}\|_p\beta_1 +\|C_2\exp(A_2\cdot(\cdot))\|_p b_\xi + 2b_{w_{2,0}}$. Comparably repeating the aforementioned proof procedure for t l belong to i^{th} sub-interval, that is $t\in\Omega_i, (i = 1, 2, \cdot, q)$ and indicating $\lim_{k\to\infty}\sup\|\Delta x_k(t_{i-1})\|_p = \alpha_{i-1}$ and $\lim_{k\to\infty}\sup\|e_k(t_{i-1})\|_p = \beta_{i-1}$, In light of the inequalities, we may say

$$\lim_{k\to\infty}\sup\|e_{k+1}(\cdot)\|_p \leq \frac{\|C_i\exp(A_i\cdot(\cdot))\|_p\alpha_{i-1}+\|C_i\exp(A_i\cdot(\cdot))D_i\|_p\gamma_{i-1}}{1-\rho_i} + \frac{\|C_i\exp(A_i\cdot(\cdot))B_i\Gamma_{d,i}\|_p\beta_{i-1}+\|C_i\exp(A_i\cdot(\cdot))\|_p b_\xi + 2b_{w_{i,0}}}{1-\rho_i}$$

$$= \frac{\delta_i}{1-\rho_i}, \tag{3.15}$$

satisfied on the time sub-interval $\Omega_i, (i = 1, 2, \cdots, q)$, where $\delta_i = \|C_i\exp(A_i\cdot(\cdot))\|_p\alpha_{i-1}+\|C_i\exp(A_i\cdot(\cdot))D_i\|_p\gamma_{i-1} +\|C_i\exp(A_i\cdot(\cdot))B_i\Gamma_{d,i}\|_p\beta_{i-1}+\|C_i\exp(A_i\cdot(\cdot))\|_p b_\xi + 2b_{w_{i,0}}$. In other words, throughout successive time intervals from Ω_1 to Ω_q, the output can converge into a neighborhood of the targeted or reference or desired output trajectory $y_d(t)$, and it also does for the whole time period Ω. This proof is complete. □

Remark 3.3. If $x_k(t-\tau) = 0$ and $\xi_k(0) = 0, \forall k \in \mathbb{N}$, then result become same as in [18].

4. Conclusion

The impact of traditional PD-type ILC on the LCTDSS with state uncertainties and observation noise has been examined in this study. The findings demonstrate that the control method is convergent, despite the fact that switching may take place at any instant when noise is present, and resilience may be ensured in the presence of bounded noise. We examine the impact of environmental noise and state time delay on tracking performance. There is also the option to analyze different ILC types for systems with many inputs and outputs that have a nonlinear continuous time delay.

5. Conflicts of interest

There are no conflicts to declare.

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APPROXIMATION OF FUNCTION IN BESOV SPACE USING EULER HAUSDORFF PRODUCT MEANS

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Abstract

In this paper, we study the degree of approximation of function in Besov space using Euler Hausdorff product means of Fourier Series and we also deduce some corollaries of our main result.
2020 Mathematical Sciences Classification: 41A10 , 41A25 , 42B05 , 42A50
Keywords and Phrases: Degree of approximation, Besov space, Euler mean, Hausdorff mean, Euler Hausdorff mean.

1 Introduction

In the last few decades several researchers have studied the degree of approximation of function in Lipschitz class and Hölder space has been studied by [1,2,3,6,7] using different product summability means of Fourier series on Conjugate Fourier also. Besov space describes the smoothness properties of functions and contain many fundamental spaces such as Lipschitz space, Hölder space, etc. Mohanty *et al.* [4], Mohanty *et al.* [5], Nigam *et al.* [8] studied the approximation function in Besov space by various summability means of their Fourier series. In the present work , we obtain the degree of approximation of function in Besov space using Euler Hausdorff product means.

2 Definitions and Notations

Let $C_{2\pi} = C[0, 2\pi]$ denotes the Banach space of all 2π- periodic continuous functions f defined on $[0, 2\pi]$ under the sup norm, and

$$L_p = L_p[0, 2\pi] = \{f : [0, 2\pi] \to \mathbf{R}; \int_0^{2\pi} |f(x)|^p dx < \infty\}, p \geq 1,$$

be the space of all 2π- periodic integrable functions. The L_p- norm of function f is defined by

$$\|f\|_p := \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p dx\right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ ess\sup_{0<x\leq 2\pi} |f(x)|, & p = \infty. \end{cases}$$

The k^{th} order modulus of smoothness of signal $f \in L_p, 0 < p \leq \infty$ is defined by

$$\omega_k(f,t)_p = \sup_{0<h\leq t} \left\|\nabla_h^k(f,\cdot)\right\|_{p'}$$

where $\delta_h^k(f,x) = \sum_{i=0}^k (-1)^{k-i}\binom{k}{i} f(x+ih), k \in \mathbf{N}$. For $p = \infty, k = 1$ and a continuous function f, the modulus of smoothness $\omega_k(f,t)_p$ reduces to the modulus of continuity $\omega(f,t)$ also for $0 < p < \infty$ and $k = 1$ $\omega_k(f,t)_p$ becomes the integral modulud of continuity $\omega(f,t)_p$.

2.1 Lipschitz Space

If a function $f \in C_{2\pi}$ and $\omega(f,t) = O(t^\alpha), 0 < \alpha \leq 1$ then $f \in Lip\ \alpha$. If a function $f \in L_p$, $0 < p < \infty$ and $\omega(f,t)_p = O(t^\alpha)$, $0 < \alpha \leq 1$ then $f \in Lip(\alpha, p)$. For $p = \infty$, the *class* $Lip(\alpha,p)$ reduces to the class $Lip\ \alpha$.

Let $\alpha > 0$ be given and let k denote the smallest integer $k > \alpha$ that is, $k = [\alpha] + 1$. For $f \in L_p$, if $\omega_k(f,t)_p = O(t^\alpha), t > 0$. Then the seminorm is

$$|f|_{Lip^*(\alpha,p)} = \sup_{t>0}(t^\alpha \omega_k(f,t)_p)$$

Thus $Lip(\alpha,p) \subseteq Lip^*(\alpha,p)$.

2.2 Hölder Space

For $0 < \alpha \le p$, let $H_\alpha = \{f \in C_{2\pi} : \omega(f,t)\} = O(t^\alpha)$. It is known that H_α is a Banach space with norm

$$\|f\|_\alpha = \|f\|_c + \sup_{t>0}(t^{-\alpha}\omega(t)), \text{ and } \|f\|_0 = \|f\|_c$$

and $H_\alpha \subseteq H_\beta \subseteq C_{2\pi}$ for $0 < \beta \le \alpha \le 1$. The metric induced by the norm $\|\cdot\|_\alpha$ on H_α is call the Hölder metric.

For $0 < \alpha \le 1$ and $0 < p \le \infty$, let

$$H_{\alpha,p} := H_{\alpha,p}[0,2\pi] = \{f \in C_{2\pi} : \omega(f,t)_p = \circ(t^\alpha)\}$$

with the norm $\|\cdot\|_{\alpha,p}$ defined as follows:

$$\|f\|_{\alpha,p} = \|f\|_p + \sup_{t>0}(t^{-\alpha}\omega(f,t)_p), \text{ for } 0 < \alpha \le 1 \text{ and } \|f\|_{0,p} = \|f\|_p$$

then $H_{\alpha,p}$ is a Banach space for $p \ge 1$ and a complete p-normed space for $0 < p < 1$.

For

$$H_{\alpha,p} \subseteq H_{\beta,p} \subseteq L_p, \text{ for } 0 < \beta \le \alpha \le 1.$$

2.3 Besov Space

Let $\alpha > 0$ be given, and let $k = [\alpha] + 1$. For $0 < p, q \le \infty$, the Besov space $B_q^\alpha(L_p)$ is the collection of all the 2π- periodic function $f \in L_p$ such that

$$|f|_{B_q^\alpha(L_p)} := \|\omega_k(f,\cdot)\|_{\alpha,q} = \begin{cases} \left(\int_0^\pi [t^{-\alpha}\omega(f,t)_p]^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 0 < q < \infty \\ \sup_{t>0}(t^{-\alpha}\omega(f,t)_p), & q = \infty \end{cases}$$

is finite. It is known that above relation is a semi-norm if $1 \le p, q \le \infty$, and a quasi-norm in other case. The quasi-norm for $B_q^\alpha(L_p)$ is

$$\|f\|_{B_q^\alpha(L_p)} := \|f\|_p + |f|_{B_q^\alpha(L_p)} = \|f\|_p + \|\omega_k(f,\cdot)\|_{\alpha,q}.$$

For $q \ne \infty$, $B_q^\alpha(L_p) = Lip^*(\alpha, p)$. When $0 < \alpha < 1$, the space $B_q^\alpha(L_p)$ reduce to $H_{\alpha,p}$ and we take $p = q = \infty$ and $0 < \alpha < 1$, the besov space reduce to the H_α.

We write through the paper

$$\varphi(x,t,u) = \begin{cases} \varphi_{x+t}(u) - \varphi_x(t), & 0 < \alpha < 1 \\ \varphi_{x+t}(u) + \varphi_{x-t}(u) - 2\varphi_x(u), & 0 \le \alpha < 2. \end{cases}$$

Theorem 2.1. *The Hausdroff matrix summability transform of $s_k(f;x)$ by $t_n^H(x)$, we get*

$$t_n^H(x) = \sum_{k=0}^n h_{n,k} s_k(f;x).$$

The (E,q) transform of t_n^H denoted by K_n^{EH} is given by

$$K_n^{EH} = (1+q)^{-n} \sum_{k=0}^n \binom{n}{k} q^{n-k} \sum_{v=0}^k h_{n,k} s_k(f;x)$$

and

$$M_n(u) = \frac{(1+q)^{-n}}{2\pi} \sum_{k=0}^n \binom{n}{k} q^{n-k} \sum_{v=0}^k \int_0^1 \binom{k}{v} z^v (1-z)^{k-v} d\alpha(z) \frac{\sin\left(v+\frac{1}{2}\right)u}{\sin\frac{u}{2}} du.$$

3 Main Theorem

Let f be 2π-periodic functions and Lebsesgue integrable for $0 \le \beta < \alpha < 2$. The best error approximation of f in the Besov space $B_q^\alpha(L_p)$ $p \ge 1, 1 < q \le \infty$ by K_n^{EH} transform of its Fourier series is given by

$$E_n(f) = \left\|T_n^{EH}(\cdot)\right\|_{B_q^\alpha(L_p)} = O(1) \begin{cases} (n+1)^{-1}, & \alpha - \beta - q^{-1} > 1 \\ (n+1)^{-\alpha+\beta+q^{-1}}, & \alpha - \beta - q^{-1} < 1 \\ (n+1)^{-1}[\log(n+1)\pi]^{1-q^{-1}}, & \alpha - \beta - q^{-1} = 1. \end{cases}$$

φ

4 Lemmas

We need following lemmas in the proof of our main result.

Lemma 4.1. *([1])* $|M_n(u)| = O(n+1)$, *for* $0 \le u \le \frac{1}{(n+1)}$.

Lemma 4.2. *([1])* $|M_n(u)| = O((n+1)^{-1}u^{-2})$ *for* $\frac{1}{(n+1)} \le u \le \pi$.

Lemma 4.3. *([4]) Let* $1 \le p \le \infty$, *and* $0 < \alpha < 2$. *If* $f \in L_p$ *then for* $0 < t,\ u \le \pi$
(i) $\|\varphi(\cdot, t, u)\|_p \le 4\omega_k(f, t)_p$
(ii) $\|\varphi(\cdot, t, u)\|_p \le 4\omega_k(f, u)_p$,
(iii) $\|\varphi(u)\| \le 2\omega_k(f, u)_p$,
where $k = [\alpha] + 1$.

Lemma 4.4. *([4]) Let* $0 < \beta < \alpha < 2$. *If* $f \in B_q^\alpha(L_p)$, $p \ge 1, 1 < 1 < \infty$, *then*

$$\int_0^\pi |M_n(u)| \left(\int_0^u \frac{\|\varphi(\cdot, t, u)\|_p^q}{t^{\beta q}} \frac{dt}{t} \right)^{\frac{1}{q}} = O(1) \left\{ \int_0^\pi (u^{\alpha-\beta}|M_n(u)|)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

$$= O(1) \left\{ \int_0^\pi \left(u^{\alpha-\beta+\frac{1}{q}}|M_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}.$$

Lemma 4.5. *([4]) Let* $0 \le \beta < \alpha < 2$ *and* $f \in B_q^\alpha(L_p), p \ge 1, q = \infty$ *then*

$$\sup_{0<t,u\le\pi} (t^{-\beta} \|\varphi(\cdot, t, u)\|_p) = O(u^{\alpha-\beta}).$$

Lemma 4.6. *(i)* $N_n(y, t) = \int_0^\pi M_n(u)\phi(y, t, u)$,
(ii) $\omega_k(T_r, t) = \|N_n(\cdot, t)\|_p$.

5 Proof of the Main theorem

5.1 Case I:

For $1 < q < \infty$, $p \ge 1$, $0 \le \beta < \alpha < 2$.

Proof. We have

$$s_k(f; x) - f(x) = \frac{1}{2\pi} \int_0^\pi \varphi(x, t) \frac{\sin\left(k + \frac{1}{2}\right)t}{\sin\frac{t}{2}} dt.$$

The Hausdorff matrix summability transform of $s_k(f; x)$ by $t_n^H(x)$, we get

$$\begin{aligned} t_n^H(x) - f(x) &= \sum_{k=0}^n h_{n,k}\{s_k(f; x) - f(x)\} \\ &= \frac{1}{2\pi}\varphi(x, t) \sum_{k=0}^n \binom{n}{k} \Delta^{n-k} \left(\int_0^1 z^k d\alpha(z) \right) \frac{\sin\left(k + \frac{1}{2}\right)t}{\sin\frac{t}{2}} dt \\ &= \frac{1}{2\pi}\varphi(x, t) \sum_{k=0}^n \int_0^1 \binom{n}{k} z^k (1-z)^{n-k} d\alpha(z) \frac{\sin\left(k + \frac{1}{2}\right)t}{\sin\frac{t}{2}} dt. \end{aligned} \tag{5.1}$$

The (E, q) transform of t_n^H denoted by K_n^{EH} is given by

$$\begin{aligned} K_n^{EH} - f(x) &= (1+q)^{-n} \sum_{k=0}^n \binom{n}{k} q^{n-k} \{t_n^H(x) - f(x)\} \\ &= (1+q)^{-n} \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{2\pi} \int_0^\pi \varphi(x, t) \sum_{v=0}^k \int_0^1 \binom{k}{v} z^v (1-z)^{k-v} d\alpha(z) \frac{\sin\left(v + \frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} dt \right\} \\ &= (1+q)^{-n} \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{2\pi} \int_0^\pi \varphi(x, u) \sum_{v=0}^k \int_0^1 \binom{k}{v} z^v (1-z)^{k-v} d\alpha(z) \frac{\sin\left(v + \frac{1}{2}\right)u}{\sin\left(\frac{u}{2}\right)} du \right\} \end{aligned}$$

(Replacing t by u)

$$= \int_0^\pi \varphi_x(u) M_n(u) dt. \tag{5.2}$$

Let

$$T_n(x) = K_n^{EH}(x) - f(x) = \int_0^\pi \varphi(x,u) M_n(u) dt. \tag{5.3}$$

Using the definition of Besov norm, we have

$$\|f\|_{B_q^\alpha(L_p)} := \|f\|_p + |f|_{B_q^\alpha(L_p)} = \|f\|_p + \|\omega_k(f,\cdot)\|_{\alpha,q}.$$

$$\|T_n(\cdot)\|_{B_q^\beta(L_p)} = \|T_n(\cdot)\|_p + \|\omega_k(T_n,\cdot)\|_{\beta,q}. \tag{5.4}$$

Using Lemma 4.3(iii), we get

$$\|T_n(\cdot)\|_p \leq \int_0^\pi \|\varphi(u)\|_p |M_n(u)| du \leq \int_0^\pi 2\omega_k(f,u)_p |M_n(u)| du. \tag{5.5}$$

□

Employing Hölder inequality, we have

$$\|T_n(\cdot)\|_p \leq 2 \left\{ \int_0^\pi \left(u^{\alpha+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \left\{ \int_0^\pi \left(\frac{\omega_k(f,u)_p}{u^{\alpha+\frac{1}{q}}} \right)^q du \right\}^{\frac{1}{q}}.$$

Making an appeal to Besov space definition , we establish

$$\|T_n(\cdot)\|_p = O(1) \left\{ \int_0^\pi \left(u^{\alpha+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}$$

$$= O(1) \left[O(1) \left\{ \int_0^{\frac{1}{n+1}} \left(u^{\alpha+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} + O(1) \left\{ \left\{ \int_{\frac{1}{n+1}}^\pi \left(u^{\alpha+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \right\} \right]$$

$$= O(1)(E+H). \tag{5.6}$$

Using Lemma 4.1 in E of (5.6), we attain

$$E = \left\{ \int_0^{\frac{1}{n+1}} \left(u^{\alpha+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

$$O \left\{ \int_0^{\frac{1}{n+1}} \left(u^{\alpha+\frac{1}{q}} (n+1) \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

$$= \left\{ (n+1)^{\frac{q}{q-1}} \int_0^{\frac{1}{n+1}} \left(u^{\alpha+\frac{1}{q}} \right) du \right\}^{1-\frac{1}{q}}$$

$$= O(n+1)^{-\alpha}. \tag{5.7}$$

Emplying Lemma 4.2 in H of (5.6), we derive

$$H = \left\{ \int_{\frac{1}{n+1}}^\pi \left(u^{\alpha+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

$$= \left\{ \int_{\frac{1}{n+1}}^\pi \left(u^{\alpha+\frac{1}{q}} \frac{1}{(n+1)u^2} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

$$= \left\{ \int_{\frac{1}{n+1}}^\pi \left(u^{\alpha+\frac{1}{q}-2} \frac{1}{(n+1)} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

$$= O(n+1)^{-1} \left\{ \int_{\frac{1}{n+1}}^\pi u^{\frac{q}{q-1}(\alpha-1)-1} du \right\}^{1-\frac{1}{q}}$$

$$O(1) \begin{cases} (n+1)^{-1}, & \alpha > 1 \\ (n+1)^{-\alpha}, & \alpha < 1 \\ (n+1)^{-1} [\log(n+1)\pi]^{1-q^{-1}} & \alpha = 1. \end{cases} \tag{5.8}$$

So, we get

$$\|T_n(\cdot)\|_p = O(1)\begin{cases}(n+1)^{-1}, & \alpha>1\\ (n+1)^{-\alpha}, & \alpha<1\\ (n+1)^{-1}[\log(n+1)\pi]^{1-q^{-1}} & \alpha=1.\end{cases} \tag{5.9}$$

By using generalized Minkowskis inequality and Lemma 4.4, we have

$$\begin{aligned}\|\omega_k(T_n,\cdot)\| &= \left\{\int_0^\pi\left(\frac{\omega_k(T_n,t)_p}{t^\beta}\right)^q\frac{dt}{t}\right\}^{\frac{1}{q}}\\ &= \left\{\int_0^\pi\left(\frac{\|N_n(\cdot,t)\|_p}{t^\beta}\right)^q\frac{dt}{t}\right\}^{\frac{1}{q}}\\ &= \int_0^\pi|M_n(u)|du\left\{\int_0^u\frac{\|\varphi(\cdot,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}}+\int_0^\pi|M_n(u)|du\left\{\int_u^\pi\frac{\|\varphi(\cdot,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}}\\ &= O(1)\left\{\int_0^\pi(u^{\alpha-\beta}|M_n(u)|^{\frac{q}{q-1}})du\right\}^{1-\frac{1}{q}}+O(1)\left\{\int_0^\pi(u^{\alpha-\beta+\frac{1}{q}}|M_n(u)|^{\frac{q}{q-1}})du\right\}^{1-\frac{1}{q}}\\ &= O(1)(E_1+H_1).\end{aligned} \tag{5.10}$$

Now, $(a+b)^r\le a^r+b^r$ for positive a,b and $0<r\le 1$ for $r=1-\frac{1}{q}<1$. we have

$$\begin{aligned}E_1 &= \left\{\int_0^\pi(u^{\alpha-\beta}|M_n(u)|^{\frac{q}{q-1}})du\right\}^{1-\frac{1}{q}}\\ &\le \left\{\left(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^\pi\right)(u^{\alpha-\beta}|M_n(u)|)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}}\\ &= E_{11}+E_{12}.\end{aligned} \tag{5.11}$$

Using Lemma 4.1, we have

$$\begin{aligned}E_{11} &= \left\{\int_0^{\frac{1}{n+1}}(u^{\alpha-\beta}|M_n(u)|)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\ &= \left\{\int_0^{\frac{1}{n+1}}(u^{\alpha-\beta}(n+1))^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\ &= O\left\{(n+1)^{-\alpha+\beta+\frac{1}{q}}\right\}.\end{aligned} \tag{5.12}$$

Using Lemma 4.2 in E_{12}, we have

$$\begin{aligned}E_{12} &= \left\{\int_{\frac{1}{n+1}}^\pi(u^{\alpha-\beta}|M_n(u)|)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\ &= \left\{\int_{\frac{1}{n+1}}^\pi\left(u^{\alpha-\beta}\frac{1}{(n+1)u^2}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\ &= O(1)\begin{cases}(n+1)^{-1}, & \alpha-\beta-\frac{1}{q}>1,\\ (n+1)^{-\alpha+\beta+\frac{1}{q}}, & \alpha-\beta-\frac{1}{q}<1,\\ (n+1)^{-1}\log[(n+1)\pi]^{1-q^{-1}} & \alpha-\beta-\frac{1}{q}=1.\end{cases}\end{aligned} \tag{5.13}$$

Combining (5.11)(5.12) and (5.13), we establish

$$E_1 = O(1)\begin{cases}(n+1)^{-1}, & \alpha-\beta-\frac{1}{q}>1,\\ (n+1)^{-\alpha+\beta+\frac{1}{q}}, & \alpha-\beta-\frac{1}{q}<1,\\ (n+1)^{-1}\log[(n+1)\pi]^{1-q^{-1}} & \alpha-\beta-\frac{1}{q}=1.\end{cases} \tag{5.14}$$

Now,

$$\begin{aligned}
H_1 &= \left\{ \int_0^{\pi} \left(u^{\alpha-\beta+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}} \\
&\leq \left\{ \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right) \left(u^{\alpha-\beta+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}} \\
&= H_{11} + H_{12}. \qquad (5.15)
\end{aligned}$$

Using Lemma 4.1 in H_{11}, we derive

$$\begin{aligned}
H_{11} &= \left\{ \int_0^{\frac{1}{n+1}} \left(u^{\alpha-\beta+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
&= \left\{ \int_0^{\frac{1}{n+1}} \left(u^{\alpha-\beta+\frac{1}{q}} (n+1) \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
&= O\{(n+1)^{-\alpha+\beta}\}. \qquad (5.16)
\end{aligned}$$

Using Lemma 4.2 in H_{12}, we obtain

$$\begin{aligned}
H_{12} &= \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(u^{\alpha-\beta+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
&= \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(u^{\alpha-\beta+\frac{1}{q}} \frac{1}{(n+1)u^2} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
&= O(1) \begin{cases} (n+1)^{-1}, & \alpha-\beta > 1 \\ (n+1)^{-\alpha+\beta}, & \alpha-\beta < 1 \\ (n+1)^{-1}[\log(n+1)\pi]^{1-q^{-1}}, & \alpha-\beta = 1. \end{cases} \qquad (5.17)
\end{aligned}$$

Combining (5.15), (5.16) and (5.17), we get

$$H_1 = O(1) \begin{cases} (n+1)^{-1}, & \alpha-\beta > 1 \\ (n+1)^{-\alpha+\beta}, & \alpha-\beta < 1 \\ (n+1)^{-1}[\log(n+1)\pi]^{1-q^{-1}}, & \alpha-\beta = 1. \end{cases} \qquad (5.18)$$

From (5.10), (5.14) and (5.18), we obtain

$$\|\omega_k(T_n,\cdot)\|_{\beta,q} = O(1) \begin{cases} (n+1)^{-1}, & \alpha-\beta-\frac{1}{q} > 1, \\ (n+1)^{-\alpha+\beta+\frac{1}{q}}, & \alpha-\beta-\frac{1}{q} < 1, \\ (n+1)^{-1}\log[(n+1)\pi]^{1-q^{-1}} & \alpha-\beta-\frac{1}{q} = 1. \end{cases} \qquad (5.19)$$

From (5.4), (5.9) and (5.19), we derive

$$\|T_n(\cdot)\|_{B_q^{\beta}(L_p)} = O(1) \begin{cases} (n+1)^{-1}, & \alpha-\beta-\frac{1}{q} > 1, \\ (n+1)^{-\alpha+\beta+\frac{1}{q}}, & \alpha-\beta-\frac{1}{q} < 1, \\ (n+1)^{-1}\log[(n+1)\pi]^{1-q^{-1}} & \alpha-\beta-\frac{1}{q} = 1. \end{cases} \qquad (5.20)$$

5.2 Case II

For $q=\infty$, $0 \leq \beta < \alpha < 2$.

$$\|T_n(\cdot)\|_{B_\infty^{\beta}(L_p)} = \|T_n(\cdot)\|_p + \|\omega_k(T_n,\cdot)\|_{\beta,\infty}. \qquad (5.21)$$

Using condition $\omega_k(f,t) = O(t^{\alpha}), t > 0$ in (5.5), we have

$$\begin{aligned}
\|T_n(\cdot)\|_p &= \int_0^{2\pi} 2\omega_k(f,u)|M_n(u)|du \\
&= O(1) \left\{ \int_0^{\frac{1}{n+1}} |M_n(u)|u^{\alpha}du + \int_{\frac{1}{n+1}}^{\pi} |M_n(u)|u^{\alpha}du \right\}
\end{aligned}$$

$$= O(1)[E_2 + H_2]. \tag{5.22}$$

Applying Lemma 4.1, we have

$$\begin{aligned} E_2 &= \int_0^{\frac{1}{n+1}} |M_n(u)| u^\alpha du \\ &\leq \int_0^{\frac{1}{(n+1)}} u^\alpha (n+1) du \\ &= (n+1)^{-\alpha}. \end{aligned} \tag{5.23}$$

Using Lemma 4.2, we derive

$$\begin{aligned} H_2 &= \int_{\frac{1}{n+1}}^{\pi} |M_n(u)| u^\alpha du \\ &\leq \frac{1}{n+1} \int_{\frac{1}{n+1}}^{\pi} u^\alpha \frac{1}{u^2} du \\ &= \begin{cases} (n+1)^{-1}, & \alpha > 1 \\ (n+1)^{-\alpha}, & \alpha < 1 \\ (n+1)^{-1}[\log(n+1)\pi], & \alpha = 1. \end{cases} \end{aligned} \tag{5.24}$$

An appeal to (5.22), (5.23) and (5.24), gives

$$\|T_n(\cdot)\|_p = O(1) \begin{cases} (n+1)^{-1}, & \alpha > 1 \\ (n+1)^{-\alpha}, & \alpha < 1 \\ (n+1)^{-1}[\log(n+1)\pi], & \alpha = 1. \end{cases} \tag{5.25}$$

Making an appeal to generalized Minkowaskis inequality and Lemma 4.6, we derive

$$\begin{aligned} \|\omega_k(T_n,\cdot)\|_{\beta,q} &= \sup_{t>0}(t^{-\beta}\omega_k(T_n,t)_p) \\ &= \sup_{t>0}(t^{-\beta}\|N_n(\cdot,t)\|_p) \\ &= \sup_{t>0}\left[t^{-\beta}\left(\frac{1}{2\pi}\int_0^{2\pi}|M_n(u)||\varphi(x,t,u)du|^p dx\right)^{\frac{1}{p}}\right] \\ &= \sup_{t>0}\left[t^{-\beta}\left(\frac{1}{2\pi}\right)^p\int_0^{2\pi}\{|M_n(u)|^p|\varphi(x,t,u)|^p dx\}^{\frac{1}{p}}du\right] \\ &= \int_0^{\pi}\left(\sup_{t>0}t^{-\beta}\|\varphi(\cdot,t,u)\|_p\right)|M_n(u)|du \\ &= O(1)\int_0^{\pi}u^{\alpha-\beta}|M_n(u)|du \\ &= O(1)\left[\left(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right)u^{\alpha-\beta}|M_n(u)|du\right] \\ &= O(1)(E_3+H_3). \end{aligned} \tag{5.26}$$

Using Lemma 4.1 in E_3, we have

$$E_3 = \int_0^{\frac{1}{n+1}} u^{\alpha-\beta}|M_n(u)|du = O\{(n+1)^{\alpha-\beta}\}. \tag{5.27}$$

Making an appeal to Lemma 4.2 in H_3, we derive

$$\begin{aligned} H_3 &= \int_{\frac{1}{n+1}}^{\pi} u^{\alpha-\beta}|M_n(u)|du \\ &= O(1)\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi} u^{\alpha-\beta-2}du \end{aligned}$$

$$= O(1)\begin{cases}(n+1)^{-1}, & \alpha-\beta>1\\ (n+1)^{-\alpha-\beta}, & \alpha-\beta<1\\ (n+1)^{-1}[\log(n+1)\pi], & \alpha-\beta=1.\end{cases} \tag{5.28}$$

An appeal to (5.26), (5.27) and (5.28) gives

$$\|\omega_k(T_n,\cdot)\|_{\beta,\infty} = O(1)\begin{cases}(n+1)^{-1}, & \alpha-\beta>1\\ (n+1)^{-\alpha-\beta}, & \alpha-\beta<1\\ (n+1)^{-1}[\log(n+1)\pi], & \alpha-\beta=1.\end{cases} \tag{5.29}$$

Emplying (5.21), (5.25) and (5.29), we establish

$$\|T_n(\cdot)\|_{B^{\beta}_{\infty}(L_p)} = O(1)\begin{cases}(n+1)^{-1}, & \alpha-\beta-\frac{1}{q}>1,\\ (n+1)^{-\alpha+\beta+\frac{1}{q}}, & \alpha-\beta-\frac{1}{q}<1,\\ (n+1)^{-1}\log[(n+1)\pi]^{1-q^{-1}} & \alpha-\beta-\frac{1}{q}=1.\end{cases} \tag{5.30}$$

6 Some Proposition

The following corollary can be derived from our main theorem.

Corollary 6.1. *The best Error approximation of f in the Besov space $B^{\alpha}_q(L_p), p \geq 1, 1 < q \leq \infty$, by $(E,q)(C,\delta)$ means of its Fourier series is given by*

$$E_n(f) = \|T_n(\cdot)\|_{B^{\alpha}_q(L_p)} = O(1)\begin{cases}(n+1)^{-1}, & \alpha-\beta-\frac{1}{q}>1,\\ (n+1)^{-\alpha+\beta+q^{-1}}, & \alpha-\beta-\frac{1}{q}<1,\\ (n+1)^{-1}\log[(n+1)\pi]^{1-q^{-1}} & \alpha-\beta-\frac{1}{q}=1.\end{cases} \tag{6.1}$$

Remark 6.1. *Corollary* 6.1 *can be further reduce in $(E,1)(C,\delta)$ means, $(E,q)(C,1)$ means and $(E,1)(C,1)$ means.*

7 Conclusion

Many researchers use various summability means to obtain the degree of approximation of functions in various spaces such as Lipschitz space, Hölder space etc. Because the Besov space generalizes to more elementary function, this space is very effective at measuring the regularity of functions. Our result generalizes many known results obtained using the Lipschitz space.

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